From Tube Maps to Neural Networks

The theory of graphs

Everything is mathematical

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Claudi Alsina

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The perfection of mathematical beauty is such that whatsoever is most beautiful and regular is also found to be the most useful and excellent.

D'Arcy Thompson

Preface

Our world does not just have letters and numbers, nowadays it is full of images too. The images that make up so much of our life fall into very distinct types. We frequently see all kinds of photographs (from memories of holidays and newspaper stories to advertising billboards), works of art of very varied styles and, along side them, less dramatic diagrams. There are diagrams in company and car logos, in traffic signs, maps, etc. Think of the diagram of your metro or bus line: a line with points that has stop names and nothing else. These diagrams with points and lines are graphs[1]. And these are the subject for this book.

We invite you to discover that, due to their extraordinary simplicity, graphs are powerful – they are schemes that allow many interesting problems to be solved and they already form a very significant part of today's mathematics.

The first chapter focuses on the birth of graph theory through a problem solved by Euler while on holiday in Prussia. You will see its rapid development during the 20th century and find out about some of its foundations. Once more familiar with the basics of graphs, you will discover some of their early uses, which will help to explain the presence of graphs in our everyday lives, where they are and what they are used for.

The second chapter covers the curious problem of colouring graphs. An apparently innocuous question like finding out what the minimum number of colours necessary for colouring a map (using different colours for countries with common borders) took from 1852 to 1976 to solve, more than a century of intense work on graphs. In the end it was confirmed that just four colours are enough, but the development of the theory over the 100-year period has been spectacular. In other words, the journey is sometimes more important than the destination.

The third chapter is dedicated to the various circuits that are of interest on a graph. Following a Chinese postman and a travelling salesman you will see that optimisation of routes, planning of timings and evaluation costs are made much easier by analysing graphs, discovering on the way why this theory served to help man land on the Moon and is today helping with the delivery of goods and tracking the construction of a building.

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1 In common parlance, the term graphics is often used to describe diagrammatic representations of events and relationships. However, mathematically speaking they are graphs.

The fourth chapter presents the surprising relationships between graphs and geometric objects. Using Euler's formula, which has been repeated over and over by students since – "Faces plus vertices equals edges plus two" – you will see how polyhedrons are studied and even how mosaics can be represented as graphs.

Finally, the last chapter covers other applications of graphs, from the Internet and technology issues to social studies, without forgetting the fun aspect of many graph-based games, which allow us to test our mental agility.

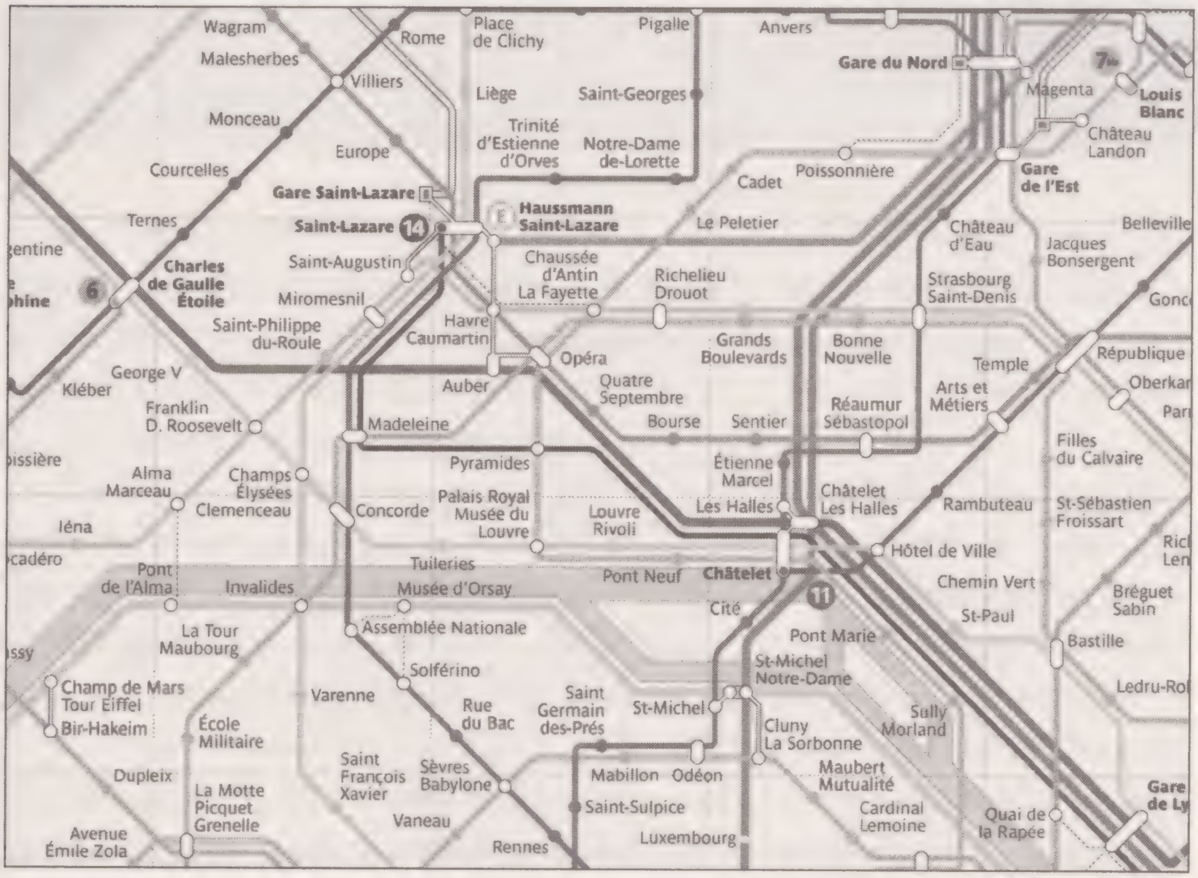
Graphs are everywhere, in science, in research, and in our personal and everyday lives. We would like this book to help you understand their importance so that you may incorporate them into your way of seeing the world. Simple things can be very useful. If they are also beautiful, so much the better.

Chapter 1

An Introduction to Graphs

Brevity is the source of wit.
Popular saying

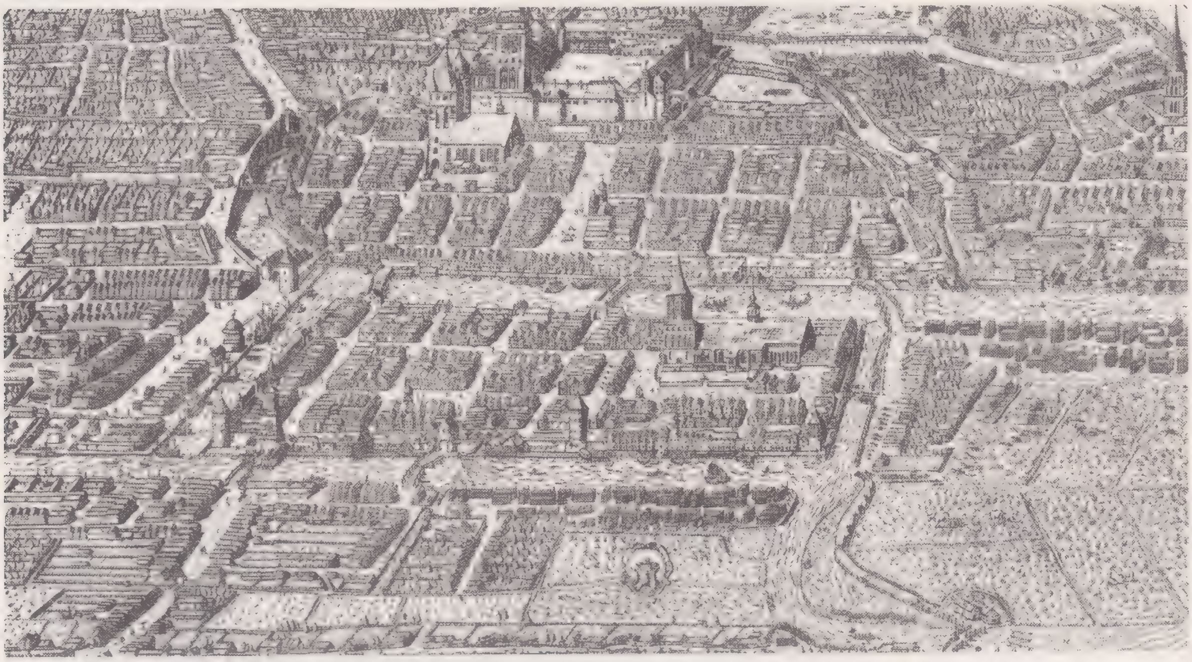
The beauty of graphic diagrams lies in their simplicity. They consist of nothing but points and the lines between them. But the most surprising thing is the power of the ideas we can express with these diagrams. This chapter invites you to begin a tour of the theory of graphs. A look at this map of the Paris Metro shows us that mathematical objects are never far removed from everyday life.



Metro maps, such as this one from Paris, are just one of many examples of graphs that we find in everyday life.

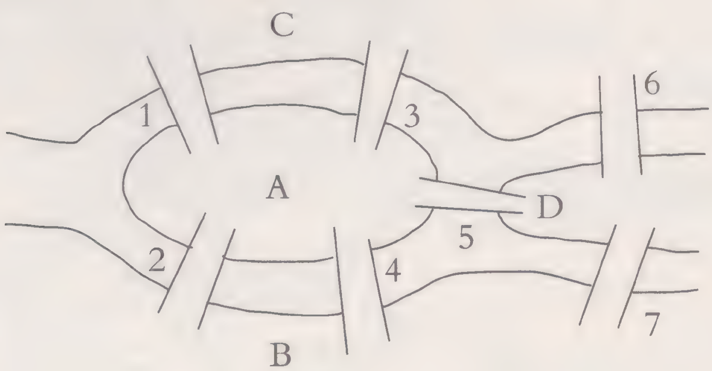
From Königsberg with love

Graph theory began thanks to a problem solved for fun by Leonhard Euler. The story goes that in 1736 our eminent mathematician stopped off during one of his tours of Europe at Königsberg (now Kaliningrad, Russia). In Euler's day, this Baltic city was divided into four parts, connected by seven bridges over the Pregel River.



The city of Königsberg in a 17th-century engraving.

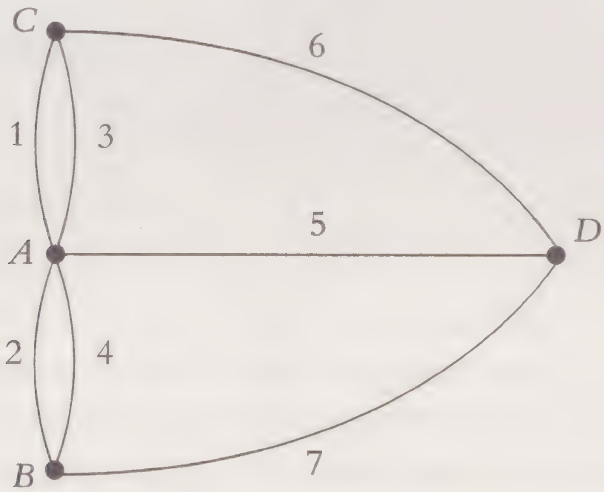
Here is simplified version of the layout, in which the bridges are numbered and each of the four city districts is given a letter:



Euler wrote: "The problem, as I understand it, is very well known and is formulated as follows: In the city of Königsberg, in Prussia, there is an island called Kneiphof, around which flow the two branches of the Pregel River. There are seven bridges

that cross the two branches of the river. The question is a matter of determining whether it is possible for a person to take a walk in such a way as to cross each of the bridges a single time. I have been told that while some denied that it was possible and others doubted it, no one maintained that it was really possible."

Euler's answer was that it was impossible and he based his negative response on the following reasoning: given the peculiar geography of the city and its surroundings, a diagram of it can be drawn using four points *A*, *B*, *C*, *D* (which correspond to the four parts of the city), and arbitrary curves connecting those point which are the real connections made by the seven bridges:



The initial problem turns out to be the equivalent to the problem shown in the above diagram. If you start at one of the four points, can you draw a route that includes all the curves just once? If that were possible the number of lines connecting each point should be even (as we will see in Chapter 3). However, all the points have an odd number of lines. Therefore, there is no solution to the problem.

The bridges of Königsberg were destroyed during World War II, but the anecdote, attributed to Euler, was the start of a highly useful and brilliant mathematical field – *graph theory*. However, it should be noted that before reaching Euler's precise formula there were many scientists who used the same concepts completely independently and only later were the ideas grouped together as graph theory.

In 1847 Kirchhoff used diagrams in the form of graphs when working on electric circuits. In 1857 Cayley studied the enumeration of isomers of an organic compound using graphs in which each point was a node with between one and four lines corresponding to chemical bonds. In 1869 Jordan studied abstract tree-like structures. In 1859 Hamilton came up with (as we will see) a polyhedron path game

LEONHARD EULER (1707–1783)

Euler was one of the greatest mathematicians of all time. Born in Switzerland, he spent a large portion of his life at the academies of St. Petersburg and Berlin. He published more than 500 papers filling 87 books. He was particularly good at algebra, number theory, geometry, mechanical analysis, astronomy and physics, and many theorems, formulae and concepts carry his name. Surprisingly, he produced more than half of his work after becoming blind in 1766. Thanks to his ideas about the bridges of Königsberg, he is considered one of the pioneers of graph theory.

that would years later lead to Hamiltonian circuits, which have numerous applications. In 1852 there was also the problem of colouring maps so that countries with a common border were different colours, and this led to a lot of research on graphs. Lewin brought this type of diagram into psychology when representing people by means of points and joining them together with lines to represent their personal relationships. Uhlenbeck, Lee and Young used diagrams with points and lines when symbolising molecular structures and their interactions.

The common factor in all pioneering cases was symbolising a specific problem by means of a graphical diagram or graph, formed by points and lines between them. Solutions to the initial problem are then arrived at by reflecting on the associations formed in the graphic. Just as completely unrelated situations can give rise to the same graphical diagrams, studying one can produce solutions to multiple problems. Of course, the creation of a graph always represents the omission of many conditions and characteristics as the graph itself must always be a simple diagram. Also note that drawing a graph is not a metric geometry problem, in other words, the lines that join the points can be of any shape. The important thing is to visualise the relationships, the connections, the interactions, not to take a photograph of a layout.

Throughout the 20th century, graph theory developed enormously both in mathematics and in its applications in all fields, finding optimum solutions to planning problems, in social sciences, architecture, engineering and particularly in computer sciences and telecommunications. Mathematically, graphs are linked to combinatorics, so-called discrete mathematics – topology, algorithm theory and knot theory.

PIONEERS OF GRAPH THEORY

Many illustrious characters have contributed to the development of graph theory, including William Thomas Tutte, Frank Harary, Edsger Wybe Dijkstra and Paul Erdös.

Briton William Thomas Tutte (1917–2002) started out as a chemist but his fondness for recreational mathematics led him to a take a further degree in that field. He began his research career in Canada in 1948. During World War II, Tutte made great contributions to deciphering encrypted German messages. His 168 articles and several books boosted the profile of graph theory and with it the study of combinatorics and discrete mathematics. Today many aspects of graphs carry his name.

American Frank Harary (1921–2005) is considered the father of modern graph theory and, in fact, he is known as Mr Graph Theory, a nickname that appears completely justified. He released 700 articles, attended conferences in 87 countries, founded the prestigious magazine *Journal of Graph Theory* in 1977, and his book *Graph Theory* from 1969 is considered the most relevant work on the subject. Harary applied graph theory not just to mathematics and computer science, but also to a range of fields from anthropology, geography and linguistics to art, music, physics, engineering, operations research... The list goes on.

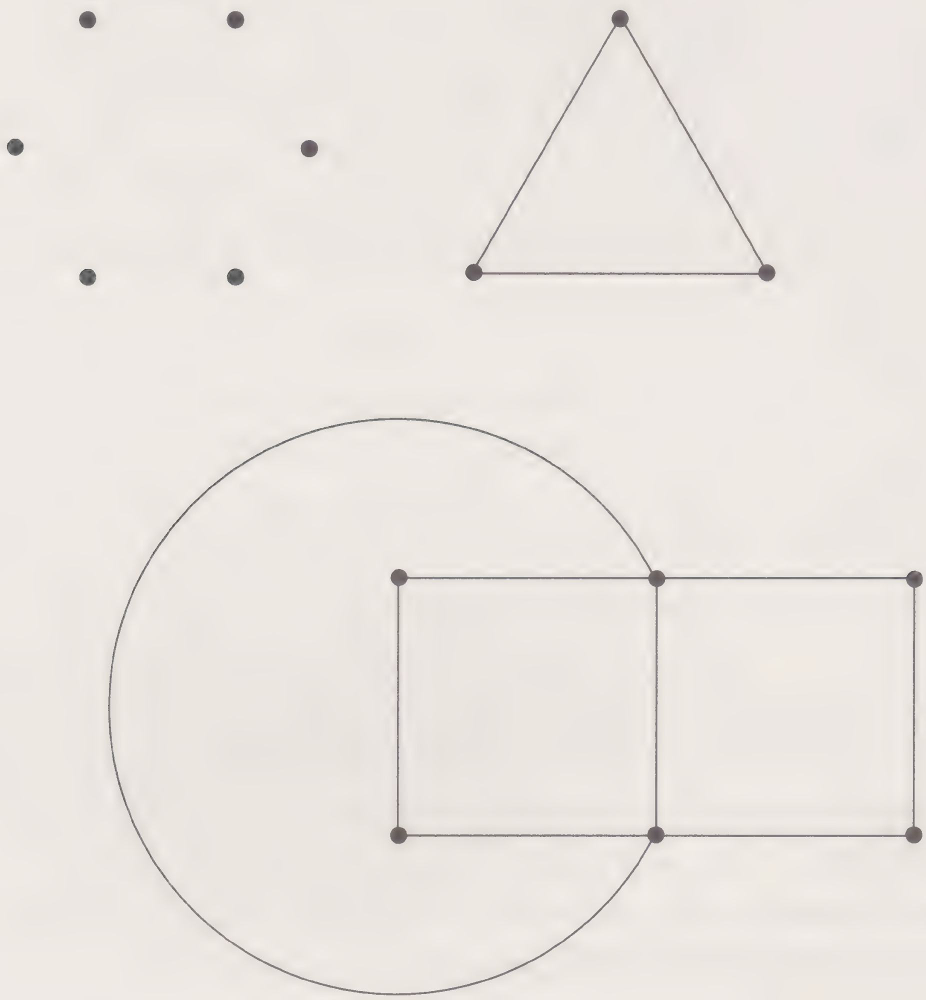
Dutchman Edsger Wybe Dijkstra (1930–2002) took a very early interest in computer programs and worked in the field for his whole life, first in the Netherlands and then moving to the University of Texas, Austin, in 1970. In 1972 he won the prestigious A.M. Turing Award for his fundamental contributions to programming languages. He is credited with the pithy phrase "Computer Science is no more about computers than astronomy is about telescopes." Dijkstra never used a computer (except to send *e-mails* and look things up on the Internet), always writing his work on algorithms and languages by hand!

Paul Erdös (1913–1996) was born in Budapest but worked in many countries throughout his life, often collaborating with others. His extraordinary intelligence allowed him to excel in graph theory, combinatorics, geometry and number theory, fields in which he was able to create magnificent problems and conjectures, writing more than 1,500 articles. According to Erdös, God must have had a book containing all of the most beautiful mathematical proofs.

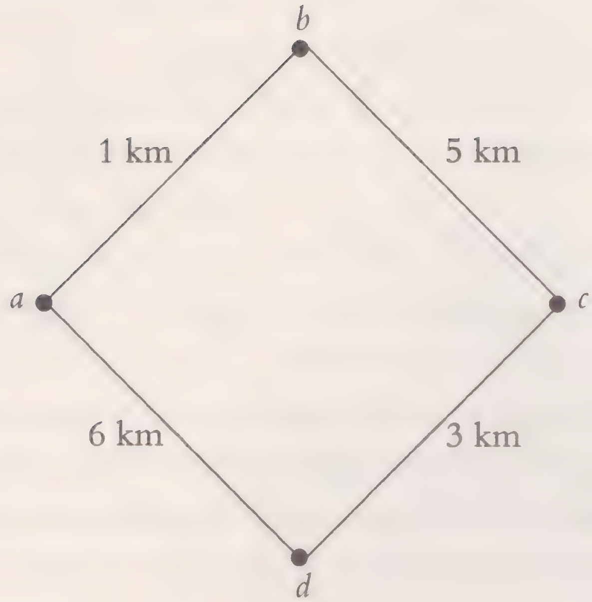
Many theories have allowed the development of graphs and in turn these have provided interesting artillery for resolving problems in other fields.

The ABC of graph theory

A graph is determined by a group of *points* (variously called elements, vertices or nodes) and by a group of *edges* or lines that join pairs of vertices.



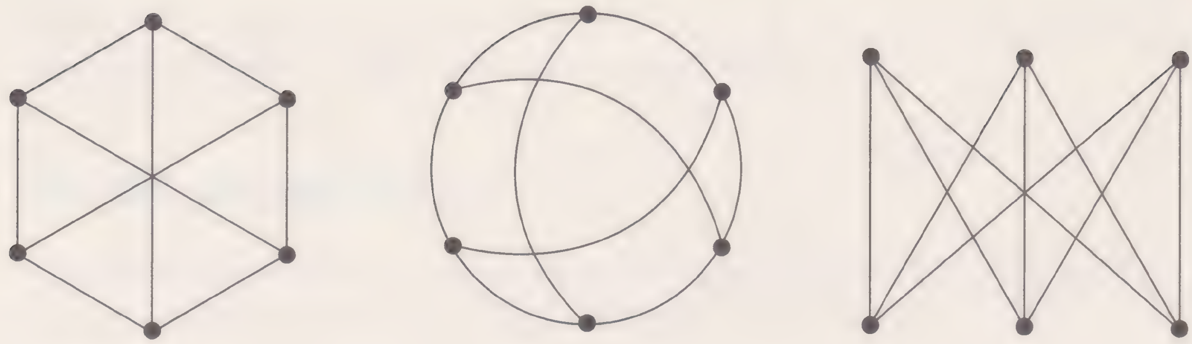
The three figures above show clockwise from top left, a *null graph* formed only by vertices; a *complete graph* formed by three vertices and their three edges, and a graph with six vertices and eight edges. Two vertices linked by one edge are called *adjacent*, two edges that share a vertex are called *incident*. The *degree* of a vertex is the number of edges connected to it.



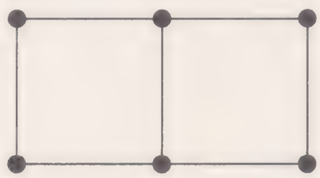
If letters, numbers, data, weights, etc. are added to the graph it becomes a *labelled and weighted graph,* and if arrows are added to the edges to indicate polarity, the directed edges have become *arcs*. When all the edges are arcs it is called a *directed graph* or *digraph*.

Although graphs could be replaced with lists, tables or long expressions, the beautiful thing about them is that they create a flexible representation. Each vertex can be represented by a *point*, a circle, a rectangle, etc. and each edge by a *continuous line* that joins the corresponding vertices (the line can be a simple segment or an artistic curve). Given this flexibility of representation, it is important to find out when two representations are *equivalent (isomorphic)*. They must represent the same vertices and show the same connections between them, in other words, there must be a two-way correspondence between the vertices and the respective edges in such a way that the correspondence preserves all degrees of the vertices.

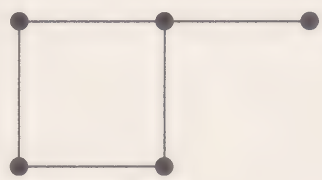
The three figures overleaf correspond to three different representations of the same graph. You have to look hard in order to see it!



The figures below show four graphs (a), (b), (c) and (d). The reference graph is (a) and the rest are *subgraphs* of (a). To form a subgraph, some of the vertices and some of the corresponding edges are used. This concept is important as it allows graphs to be studied in sections.

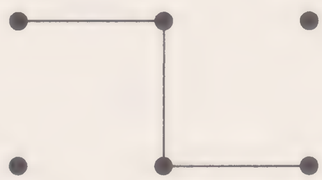


(a)



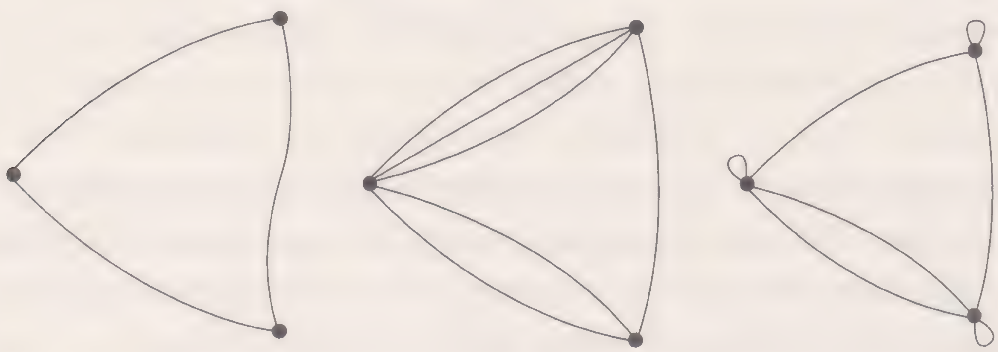
(b)

(c)



(d)

It is quite common to distinguish between three types of graph: graphs themselves, multigraphs and pseudographs. The following figures are, from left to right, an example of each one.



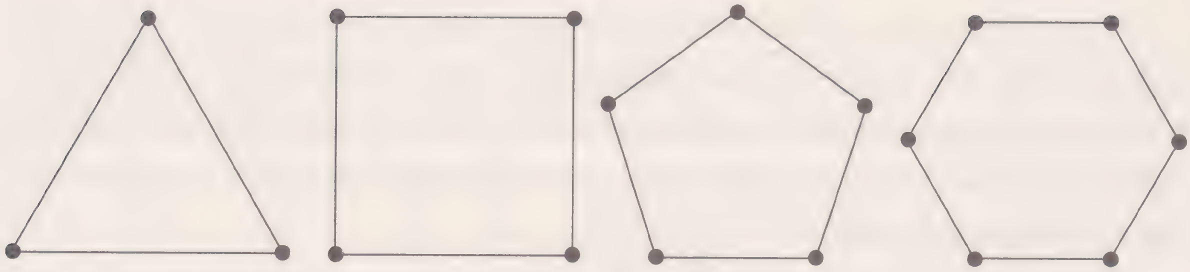
If two vertices can only be connected by one edge, it is a *graph;* if it can be done by more than one edge it is a *multigraph,* and if, on a multigraph, a vertex can be connected to itself (the self-joining edge is said to form a *loop),* then it is a *pseudograph.* In this book all of the three types are called graphs and in each specific case any restrictions are qualified.

With regards to specific paths that can be made in a graph, we use the following nomenclature. If G is a graph labelled with vertices V_0, V_1, V_2,... and edges X_1, X_2, X_3..., a *route* on G is a finite succession of the type $V_0, X_1, V_1..., V_{n-1}, X_n, V_n$, where the vertices and edges are joined. If we write (V_0, V_1..., V_n) it is understood that there is only one edge between two vertices and that the determined route is taken through those vertices. If $V_0 = V_n$, in other words, the start vertex is the same as the end vertex, the route is considered *closed,* otherwise it is called, *open.* A *path* is a route where each edge is only travelled once. A closed path with *n* different points or vertices is called a *cycle.* Note that any cycle can be represented graphically by a polygon, as we will see overleaf.

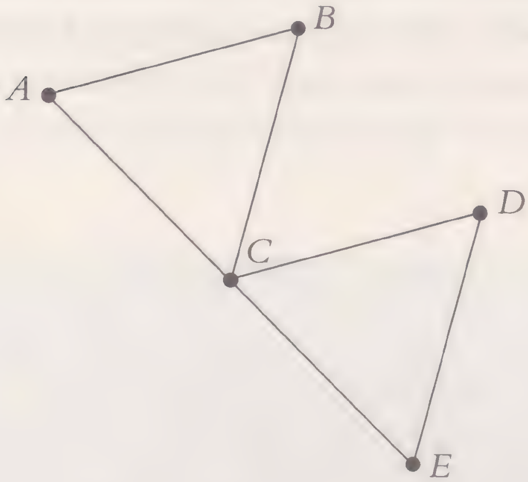
GRAPH OR GRAPHICS

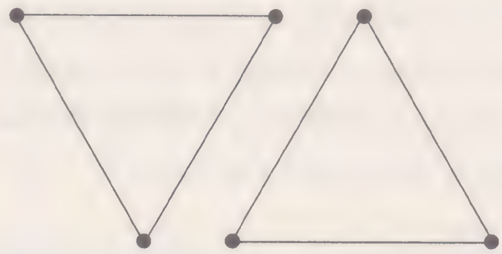
Graph is a multipurpose word, as it can mean 'writing' (as in graphology, graphomania...), indicating something that writes or records (seismograph) or something that is written. It can also refer to a type of mathematical and statistical diagram. However, in the purest form of graph theory, it means discrete drawings made up of dots and lines. So, a linear graph is formed by polygonal lines, a succession of straight segments joining up points. Perhaps more familiar are the function graphs where two axes *X* and *Y* are represented, in relation to which points (x, $f(x)$), which constitute the *graph of the function* $y = f(x)$, can be seen. These are actually diagrams that associate each point *x* of the *OX* axis with point (x, $f(x)$) of the function. They are, therefore, a special type of graph.

Graphs are used to represent the relationships between elements of a finite group. For example, certain equivalences allow the elements to be classified into classes, the 'points' of the graph represent those elements and the 'lines' are drawn between related or equivalent elements (thus, if the relation is *reflexive*, in other words, all elements are related to one another, a loop is drawn). Directed graphs are also used. To order relationships, the arcs with arrows represent the 'less than' relation. The relationships between graphs and group theory are explained in detail in the Appendix.



If a route can be drawn that joins any two vertices of the graph, the graph is said to be *connected*, as in the case in the four figures above. In the case of connected graphs it makes sense to talk about the *distance* between two vertices u and v as the minimum number of edges that allow u to be linked to v.



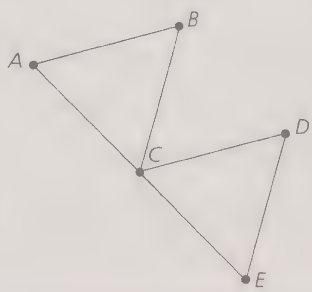


In the two examples above, the left-hand one is connected and the right-hand one is not.

GRAPHS AND NUMBERS

Graphical information on whether or not there is an edge between two vertices can be converted into numbers by means of a table or *matrix*. The graph *ABCDE* from the diagram forms the following table, into which a 1 is inserted if there is an edge and 0 if there is not.

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	0	0
C	1	1	0	1	1
D	0	0	1	0	1
E	0	0	1	1	0

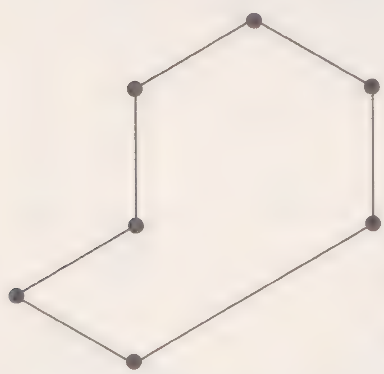


COMMUNICATION AND ERRORS

In 1956 Claude Shannon, founder of information theory, tackled the problems of distortion in communications sent via various media, or channels. The channel is modelled as a graph where the vertices are the symbols used to compose the message and the edges join the vertices that could be confused during transmission.

Polygonal and complete graphs

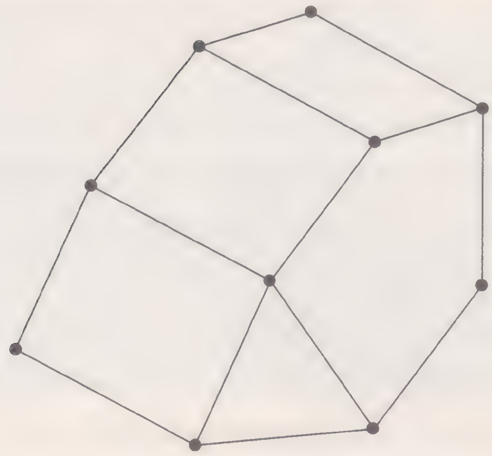
Cycles or polygons are extraordinarily simple graphs as they describe a route through all the vertices that starts and ends in the same place, as is the case in the two figures below.



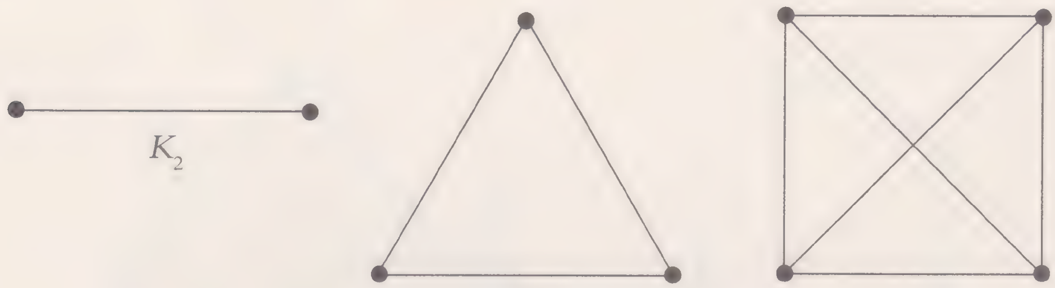
Urban bus routes or police patrols could be represented by these polygons. The number of vertices V is equal to the number of edges E.

Polygonal graphs are obtained from a combination of polygons or cycles. They have a limited number of faces, some shared vertices (except those on the edge) and various interior edges in common, as well as other edges defining the exterior. The total number of vertices V and the total number of edges E is easy to count.

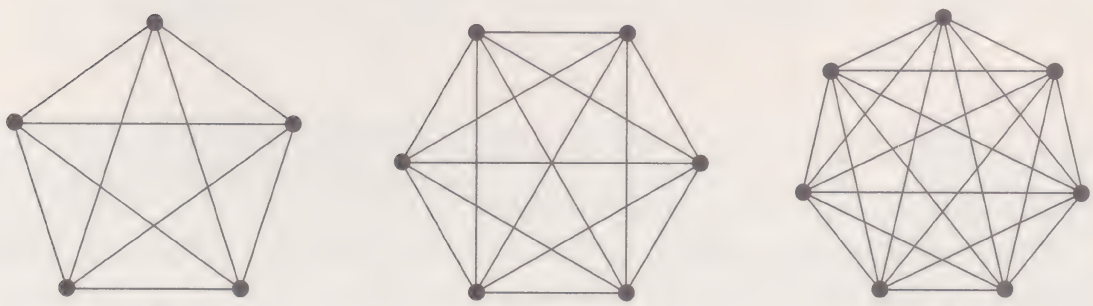
But when it comes to the number of faces F, it should be stressed that both the number of faces F themselves and the 'exterior face' are counted. For example, the following figure has 10 V, 14 E and 6 F.



Graphs in which all pairs of vertices are connected by an edge are called *complete* or universal. The following figures show examples of the first six complete graphs, from 2 to 7 vertices. If the complete graph has n vertices it is represented with the symbol K_n.



K_2 K_3 K_4



K_5 K_6 K_7

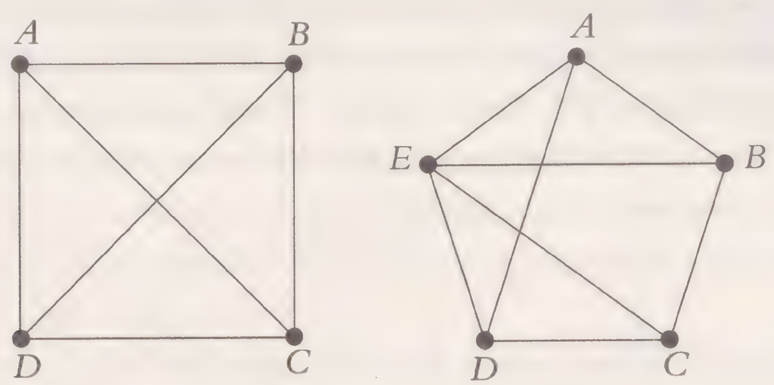
The number of edges of a complete graph K_n is very easy to count. Each vertex must be joined to the others $(n-1)$ and there are n in total $n \cdot (n-1)$ all the edges will be counted twice (as each edge has two vertices). So the total number of edges will be $n\,(n-1)/2$, which is the combinatorial number $\binom{n}{2}$ of the possible combinations of selectable pairs of a group of n elements. This quantity shows quadratic growth in n, in other words, large values for K_n have numerous edges.

TURÁN'S THEOREM

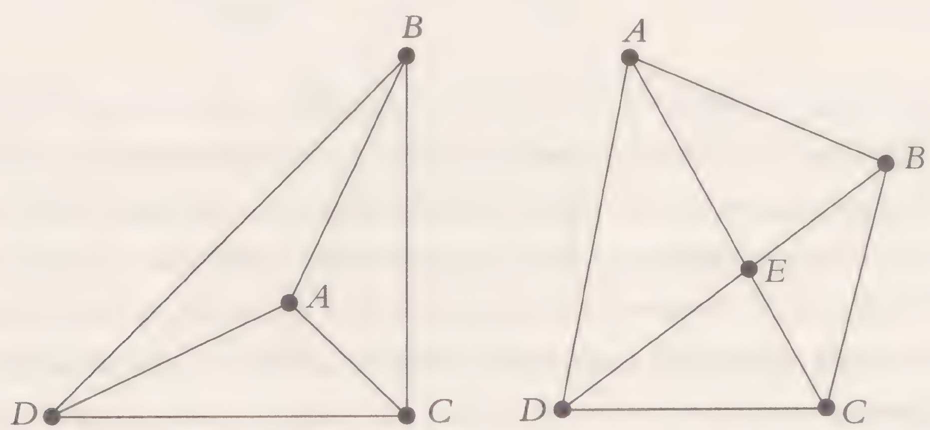
In 1941 Turán considered the following problem: Let's suppose we have a simple graph G with n vertices, given a number p ($p \geqq 2$) let's consider a *p-clique* as a complete subgraph of G with p vertices (or Kp). The question is, if the graph does not contain a *p-clique* what is the minimum number of edges the graph can have. The surprising answer is that the number of edges cannot exceed the value $n^2\ p/2\ (p-1)$. Due to a really beautiful proof, this result has become a mathematical benchmark in graph theory.

Planar graphs

Once the vertices of a graph have been drawn, placing the edges can result in drawings such as the following figures.



However, these graphs can be redrawn in a better way, maintaining the same links but with a clearer representation, in which the edges do not cross at points that are not the vertices, as in the two previous cases:



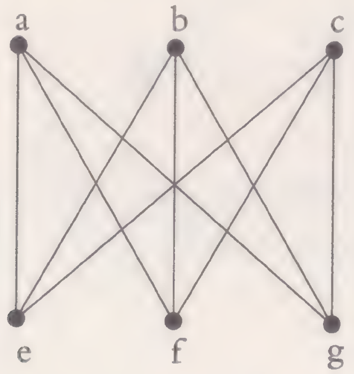
PLANAR ELEGANCE

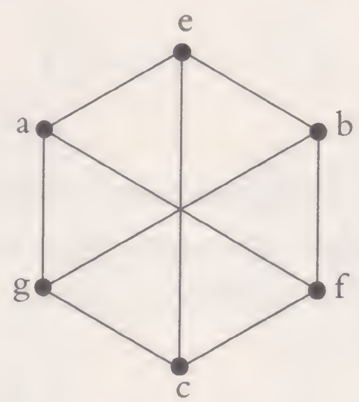
In order to draw a planar graph well there is no need to create strange edges. All planar graphs allow representation with the edges drawn with straight segments that only cross at two vertices. It could not be any more elegant.

A graph is called *planar* if it allows a representation on a plane where the edges only cross at the vertices. Note that therefore in order to analyse the 'planarity' of a graph we need to see if an equivalent (isomorph) allows such clear representation without undesirable crossing. Wouldn't it be beautiful if all graphs were planar? Before looking for an answer, let's look at a famous graph puzzle.

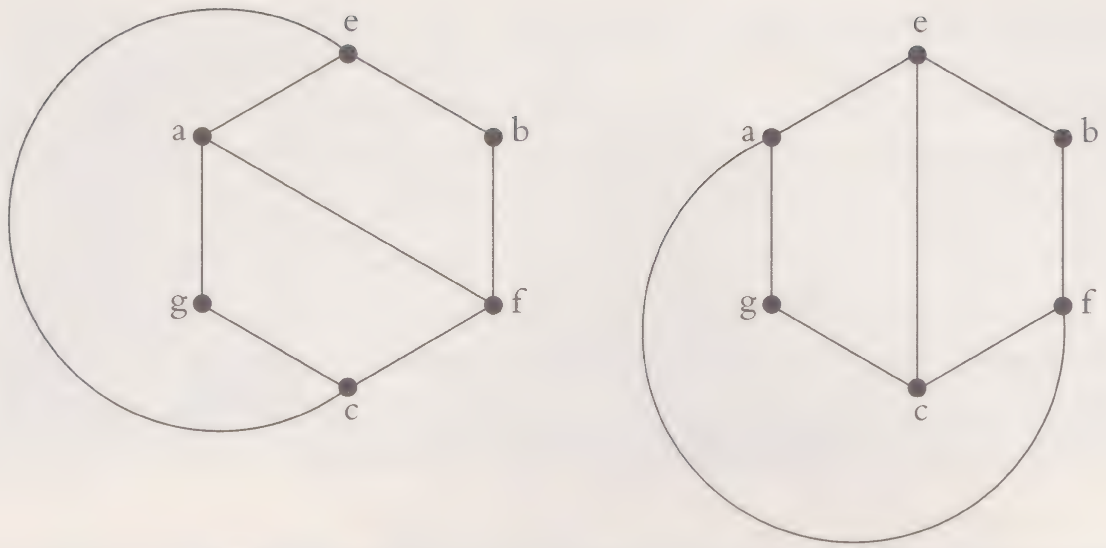
The problem of the wells and the enemy families

The problem is as follows: In three houses a, b and c there are three families who cannot stand each other. Outside there are three wells e, f and g of which one is always full and two empty. The families would like to be able to access any of the three wells, but using routes that never cross in order to avoid bumping into their neighbours. Are these nine routes possible?



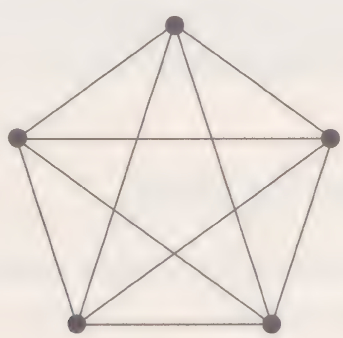


In the first figure above we can see a first attempt to join a, b and c with wells e, f and g. This type of 'three by three' graph is represented with the symbol $K_{3,3}$. This would imply numerous undesirable (and dangerous) crossovers for the enemy families. In the second figure the same graph has been drawn in a perhaps clearer form, but there are still crossovers. Notice that now if the access to well g is removed from house b, we could have eight paths without crossovers, as shown in the following two figures.



Can the missing edge be added without cutting the rest? It is pertinent to remember that the result is very intuitive (although strangely difficult to demonstrate): if a continuous, simple and plane curve divides the plane into two areas (interior and exterior), any continuous curve that joins a point from inside to another that is outside will cut at least one point of the given curve (Jordan's theorem). Looking at the above figures again we can see that in the two drawings g is inside a closed and continuous curve and b is outside of it. Therefore it is not possible to resolve the problem of the family and the three wells. The only option for family b would be to build a bridge through space to g.

The problem of the families and the wells gives us our first example of a non-planar graph – the graph previously named $K_{3,3}$ is not planar. Another simple example of a non-planar graph is complete graph K_5 of a pentagon with its pentagonal star inside, as shown in this figure:



This gets complicated at times. If graphs $K_{3,3}$ and K_5 are not planar, any other graph that extends them will not be planar either, as the cross-overs would only get worse; thus we now have infinite examples of non-planar graphs. But a pleasant

KAZIMIERZ KURATOWSKI (1896–1980)

Professor Kuratowski was one of the greatest Polish mathematicians, leading research groups and interacting with his great counterparts across the world. He worked on logic, topology, and set theory, and in 1930 surprised everyone with his famous theorem on planar graphs. Despite the practical complexity of determining the "planarity" of a graph, the wording of Kuratowski is very simple.

surprise awaits us thanks to a result from Kuratowski. Note that two graphs are called *homeomorphs* if they are identical or isomorphs when all the vertices are reduced to the second degree. The result is the Kuratowski theorem, which is as follows:

> "A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of $K_{3,3}$ or K_5."

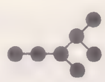
In order to see the planarity, all vertices with a degree two should be removed and then it can be seen that a $K_{3,3}$ and a K_5 are not infiltrated.

AN APPLICATION IN ARCHITECTURE

It is of interest in architecture projects to see the graph as a way of analysing the accessibility between spaces. If this graph is not planar, solutions with several floors should be sought introducing stairs. In the case of 'planarity' solutions can be found on the same floor.

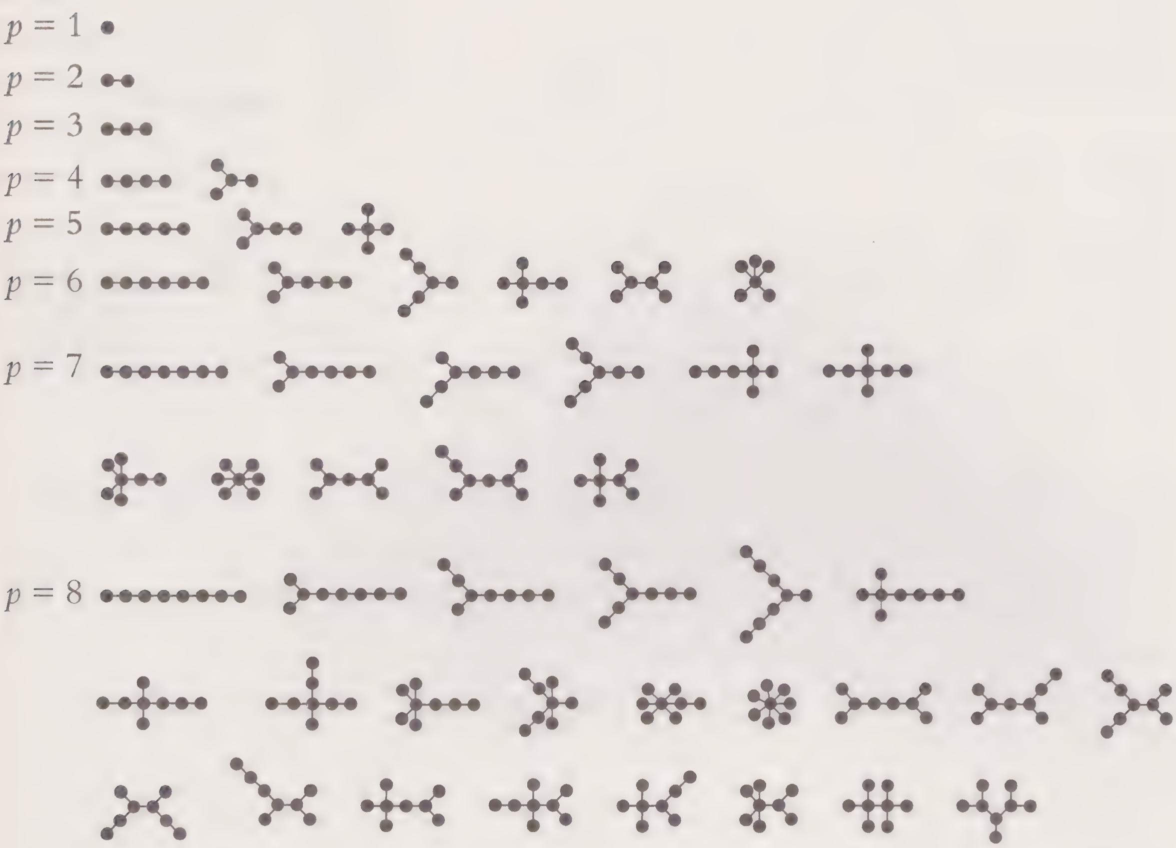
The trees do let you see the forest

A tree is a very simple graph with all the vertices connected but without any cycles or polygons, as in the following figure:



Thus, on a tree a path can be made between any two vertices.

Below are all the possible trees from 1 to 8 vertices.



The succession of numbers of possible trees for each number of vertices is: 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1,301, 3,159…

If there are p vertices then the tree always has $p-1$ edges, but for each value of p, p^{p-2} different trees can be draw (Cayley's formula). Introduced by Cayley in 1857 these constitute a very important class of graphs as they connect all the vertices using the lowest possible number of edges, which makes them useful in designing electrical circuits, telephone networks, and intercity roads networks.

The following theory, both beautiful and simple, gives the trees a characteristic that is also crucial for their applications:

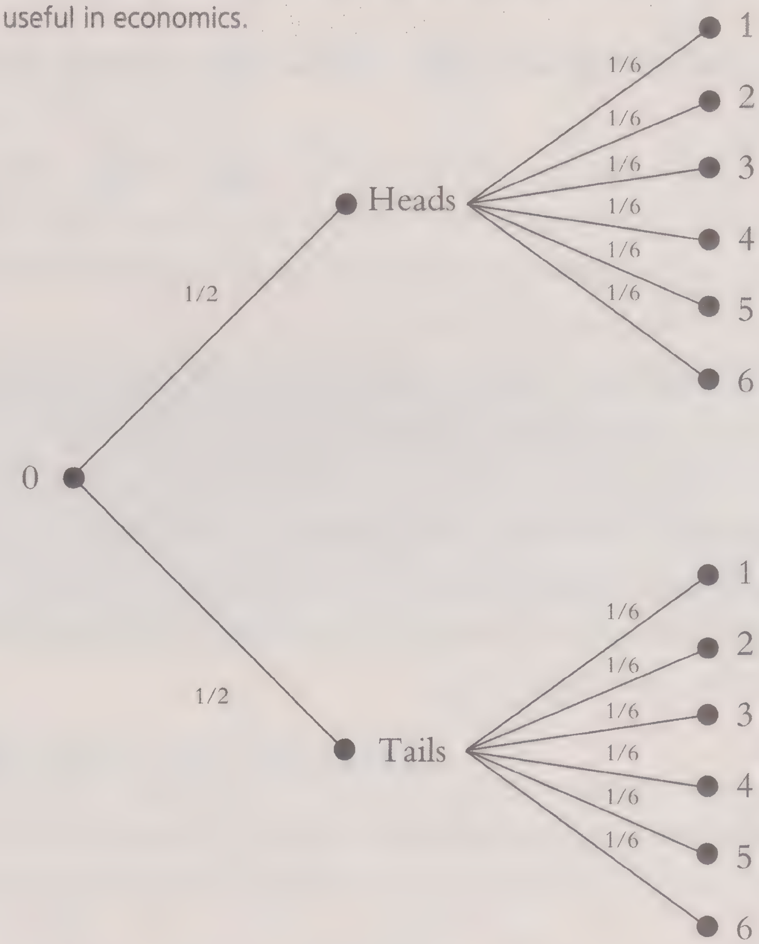
> A graph G is a tree if and only if given any two vertices, u and v of G, there is a unique route that connects u with v. This is the same as G being connected and, if it has p vertices, having $p-1$ edges.

Despite such a simple result, note in the above graphs how the number of possible trees grows enormously with p.

The argument justifying this is as follows. Take a tree, G. Given two vertices u, v of G, by virtue of the connection of G there will be at least one route between u and v. If there were two routes C_1 and C_2 between u and v, a cycle would be generated in G, which means it cannot be a tree. There can only be a single route between any two vertices for G to be a tree.

TREES AND PROBABILITIES

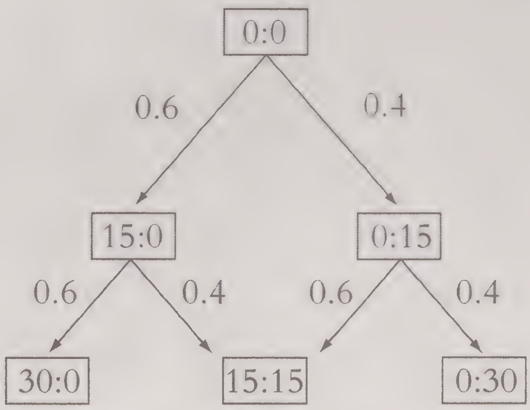
When analysing probability situations (games for example) it is common to arrange the various alternative events and their corresponding probabilities by means of a tree where the vertices correspond to events, and the values of the probabilities of the final outcome occurring once the starting move has occurred are directly placed on the edges. And the pertinent calculations are made using the tree. The figure below shows a tree that corresponds to the game of throwing a coin and then a dice. These representations are often used in game theory, which is very useful in economics.



Calculating probabilities demands clarity in order to find all possible results.

W. WINGFIELD & A.A. MARKOV: TENNIS AND GRAPHS

W. Wingfield patented a game called *tennis* in February 1874. For his part, A.A. Markov (1856–1922), through the theory of probability, considered what we today call *Markov chains*, in other words, an orientated graph, the vertices of which represent states and in whose arcs transitions from one state to another are valued, depending on the probabilities of the state of departure but not on the entire past. Wingfield and Markov were brought together by *Mathematics and Sport* a work by L.E. Sadovskii and A.L. Sadovskii, in which Markov chains are used to analyse games of tennis. For example, if the probability of the two players is valued at 0.6 and 0.4 for each point, the chain looks like this:



Let's look at a practical problem that uses Cayley's trees. Given n cities A_1, $A_2,\ldots, A_n$ and knowing all the costs that will be needed to establish a connection between each pair of cities (be it roads, water, electricity, gas, telephone, etc.), which is the cheapest network that allows all the cities to be connected by this service. Evidently, if the 'economic connection' network must be a tree, then it would require a global connection without cycles. If there is a cycle, eliminating one of its edges would make the connection cheaper.

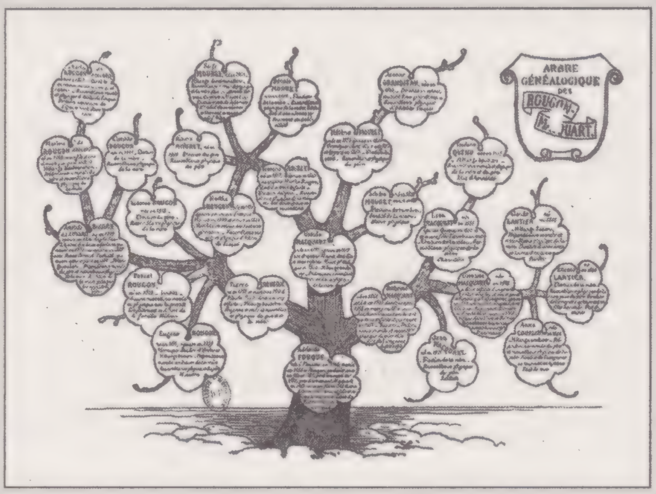
Therefore, the tree of connections between the n cities will have $n-1$ edges. First the connection is drawn between two cities that have the lowest possible cost. Then one of them is connected to a third which has the lowest connection cost, and so on.

And what is this group of tree graphs called? As you might have guessed, it is called a *forest*. In graph theory the trees let you see the forest.

GENEALOGICAL GRAPHS AND TREES

A clear way of visualising a family or person's genealogical data is to represent the relatives on a graph where photos, names and ages of direct relatives are placed on the vertices, and the son/daughter relationships are indicated on the edges. The tree can be *descending* if it illustrates all the descendants of a couple or *ascending* if it shows all the ancestors of an individual.

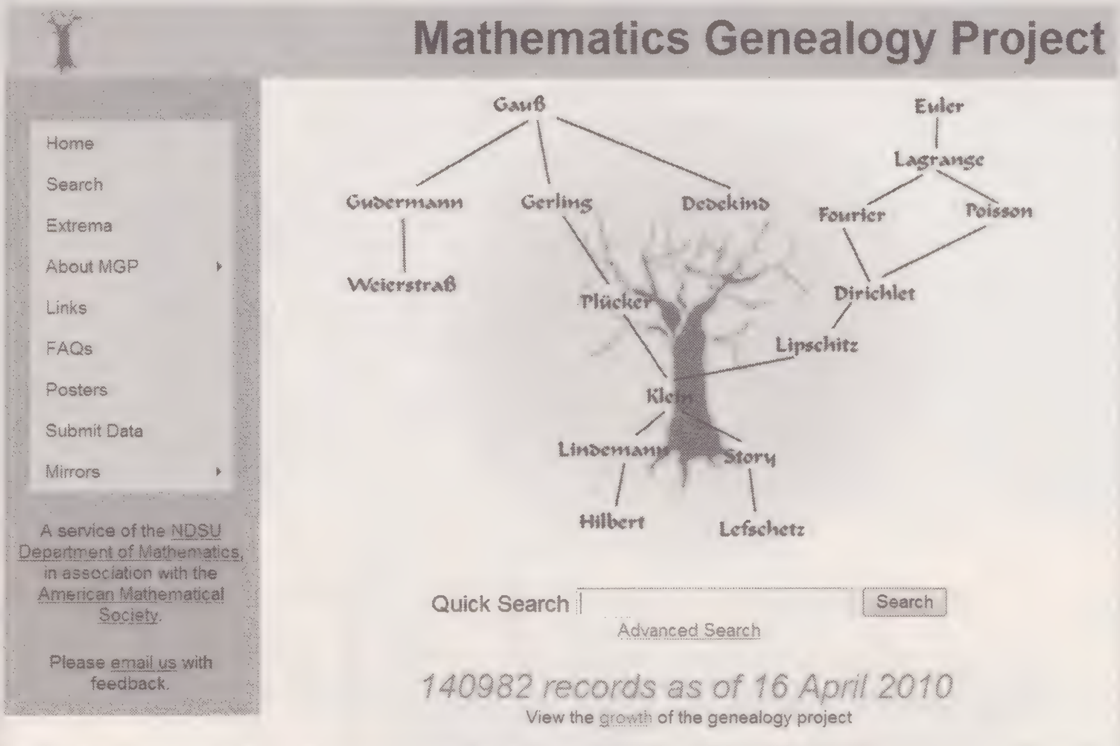
While in the past family trees were drawn with branches on which various family members were distributed, nowadays the clarity of graphs has allowed clearer and less colourful representations, many of them computerised (programs for representing family trees can be found on various websites). Today family trees are also used to represent the pedigree of dogs, race horses, bulls, related political parties, musical styles, related languages, etc.



A modern family tree created on a computer for the Romanov family and one drafted in 1878 which depicts a fictional family dreamt up by the writer Emile Zola, the Rougon-Macquarts.

A GRAPH OF MATHEMATICAL FAMILIES

The Mathematics Genealogy Project can be seen at genealogy.math.ndsu.nodak.edu. It provides huge amounts of information on mathematicians and their intellectual 'descendants', in other words, people who have done their doctorate thesis with a notable figure. At the same time, doctoral candidates of these doctoral candidates are also included, and so on. This gives a tree on research relationships of all mathematicians (in April 2010 there were already 140,982 logged).



The Mathematics Genealogy Project.

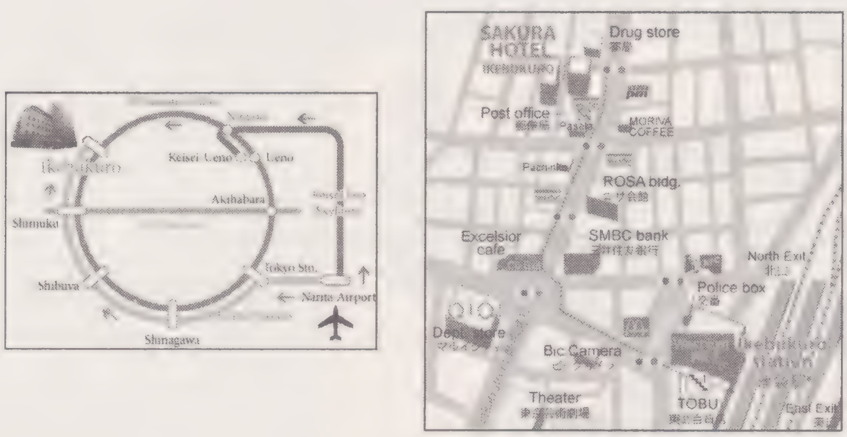
Graphs in everyday life

As well as the family trees filed away with other important documents (or the one you may decide to make after reading this chapter), there are other graph-type diagrams used in news reports to represent data on accidents per day or year, strikes, price changes, etc. Perhaps your wrist watch is a 12 point graph and, also, maybe several polygonal shapes characterise your paintings, your cutlery, your decorations, etc. The world of GPSs, maps and online route-finders offers magnificent examples of graphs: the edges are the streets, the points, places or cities; the vertices of the

graph have a name and the edges have numbers indicating the miles (a weighted labelled graph).

Road maps like this one from 1929 are a good example of a graph.

Sometimes graphic representations are even simpler. In the following figures two new forms of visual information can be seen.



A hotel in Tokyo uses various graphs to indicate how to get there from the airport.

In public transport, graphs give us well-organised information to make our journeys easier. The same graphs also help to plan new lines or extensions, new stations, etc. The New York subway map clearly indicates lines (with coloured edges), stations and possible links, although this way of drawing underground lines originates from the London Underground map. On the other hand, airlines' graphs, in which a multiplicity of lines (one for each destination) come out of one point (an airport), are much more complicated.

Public transport graphs need to be very clear, not only to indicate possible routes, but also the multiple connections that can be made.

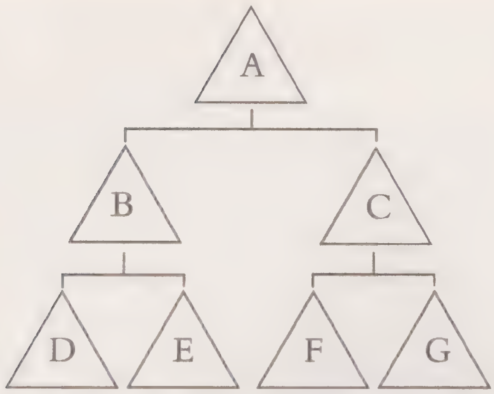
Simplicity and clarity are fundamental when it comes to drawing graphs such as the map for the New York subway, a service that handles millions of passengers a day.

A GRAPH OF THE LONDON UNDERGROUND

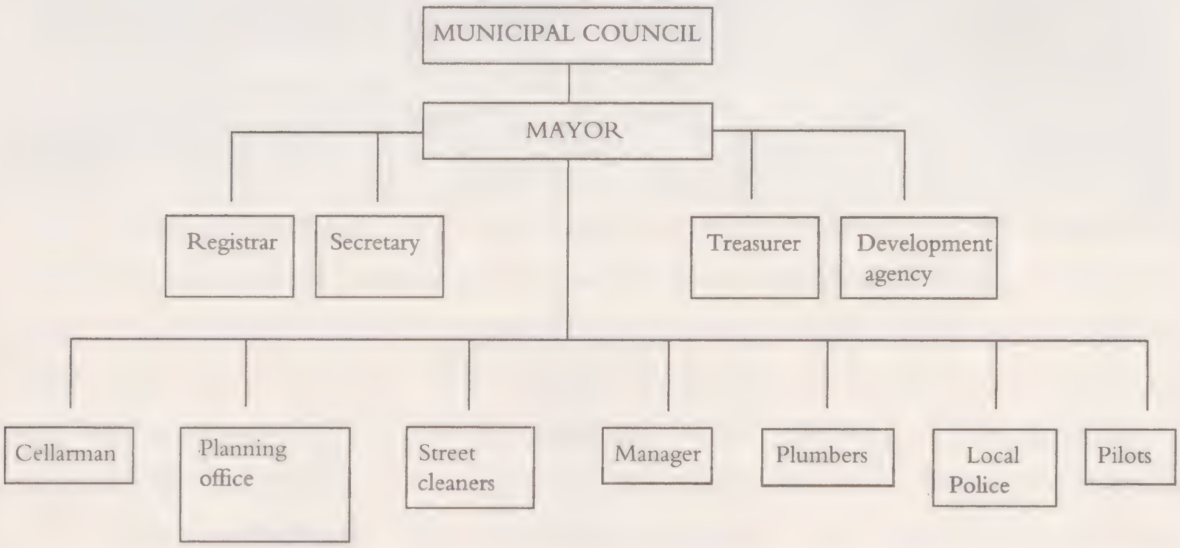
In 1909 the commercial director of the London Underground, Frank Pick, who was responsible for the company's graphics, gave various designers the job of devising an underground map to help passengers get around the system, which was becoming more and more complex. Many of the designers failed in their task as their maps were based on the various stations' location in the city, which caused great confusion among passengers as to which line (or lines) they could choose from. In the end the problem was resolved by Henry Beck (1903–1974) an engineer and designer. Beck's masterstroke was to simplify the diagrams, leaving the 'essence' of a graph with lines and communications on an octagonal grid. He placed the lines and stations in such a way that (if necessary) they formed angles of 90° or 45° lending greater visual clarity. He finished off by drawing the River Thames through the middle as means of orientating the map. Today's map still follows the same design.

There are many organisational and hierarchic diagrams, such as the one that appears opposite, which are frequently used in business presentations.

Organigrams explain dependency, possible itineraries, alternatives that could be considered, algorithms that should be followed... Order and clarity!



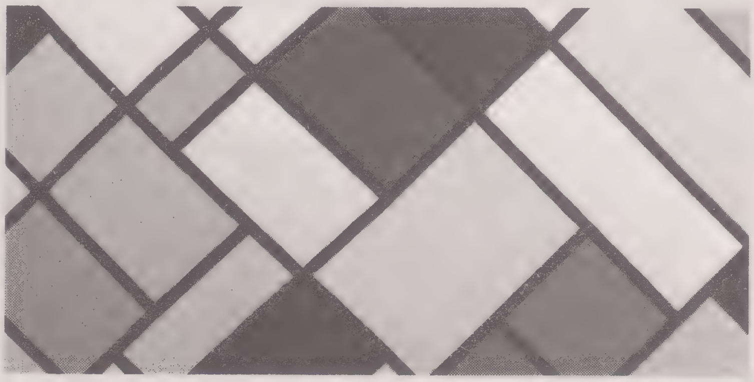
The analysis of these organigrams, such as that of the municipal council shown in the diagram below, makes the dependencies and problematic situations very clear. The example applies to a real organisation in a South American village, where the seemingly omnipotent mayor even has to take charge of a cellarman and the council's plumbers.



If you are going to make a journey you could make a graph organising it with times, costs, links, expectations... And even if you stay on your sofa to read *Treasure Island* you will see what a graph indicating where the precious treasure is hidden looks like. In the following chapters of this book, you will see how graphs can be useful in telecommunications, the Internet, to plan costs, cleaning, distribution, collections, urban developments, interior decor. They are graphs created by professionals that influence your quality of life, resulting in cleaner streets, faster mail, tidied rubbish, more comfortable homes, more habitable cities, etc.

GRAPHS AND ART

With the birth of abstract art, painters and sculptors have avoided representing people, objects and landscapes and preferred to analyse colours and shapes, corresponding to abstract shape relationships. From the Renaissance ideal of a painting as a window through which to 'see' the real world to the surrealist idea that "it is the onlooker who makes the painting" there was a sizeable paradigm shift. From the influence of written theses and paintings by Vasili Kandinski (1866–1944) or Theo van Doesburg (1883–1931), for example, the purest geometric shapes and the most basic colours were gaining strength in an art that creates emotions and beauty without necessarily being reflections of reality. Points and lines (graphs!) are thus converted into the key of artistic representation.



Counter composition XVI in dissonances, by Theo van Doesburg.

Chapter 2

Graphs and Colours

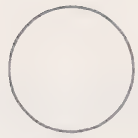
Illinois is green, Indiana is pink…
I've seen it on the map, and it's pink…
Mark Twain

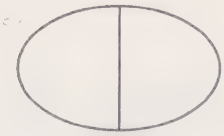
This section invites the reader to take a look at a very specific and apparently simple problem on graph theory, colouring maps. But by doing so you will discover how a seemingly trivial intellectual challenge can sometimes lead to great advances in knowledge.

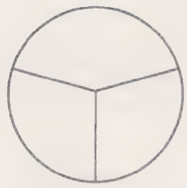
Maps and colours

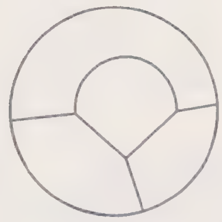
Most geographical maps can be interpreted as graphs in which the vertices are the points where three or more lines and the edges are the lines defining a territory or area. One problem faced by cartographers was trying to colour them so countries or zones always have different colours. Given the quantity of countries or regions and the limited range of colours that were used in old-fashoined colour printing, the criteria had to be modified to require that only countries with common borders had to be coloured differently. Then, naturally, the question arises: given any map, what is the minimum number of colours necessary to colour it in such a way the areas with common borders have different colours? (It is understood that the border may not be reduced to a unique point.) As the range of possible maps is huge (countries, regions, industrialised areas, population areas, etc.) it is clear that this problem must be formulated in the general context of graphs, in other words, it has to work for any map that describes a polygon graph.

To begin with we shall look at the figures overleaf. Each of them needs exactly 1, 2, 3 and 4 colours respectively to be coloured in following the rules of the game. Note that if we also want to colour in the exterior area as well, we will need 2, 3, 4 and… 4 colours.

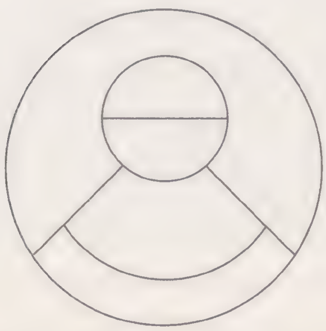

(a)

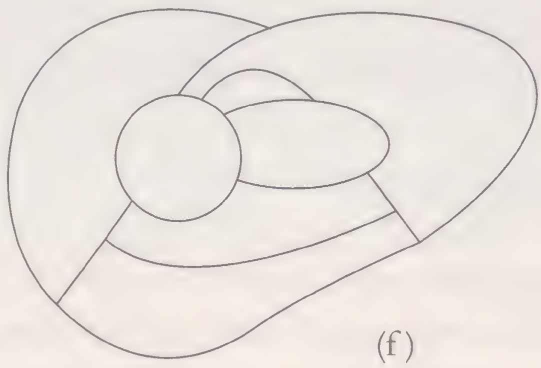

(b)


(c)


(d)

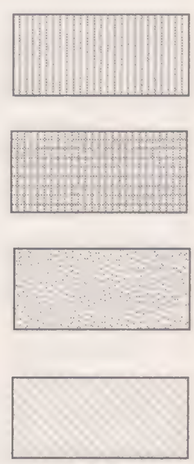
Now let's take a look at some more complex figures.

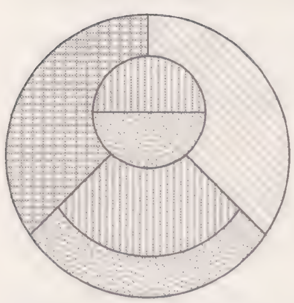

(e)

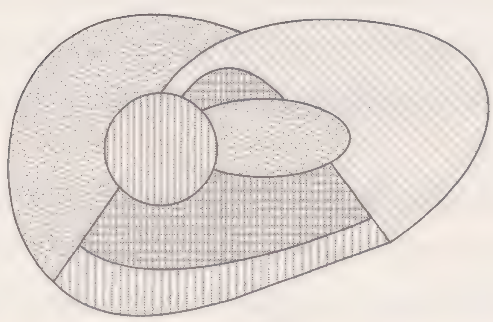

(f)

A simple attempt to colour these in shows that four colours are enough.

Colours




(g)


(h)

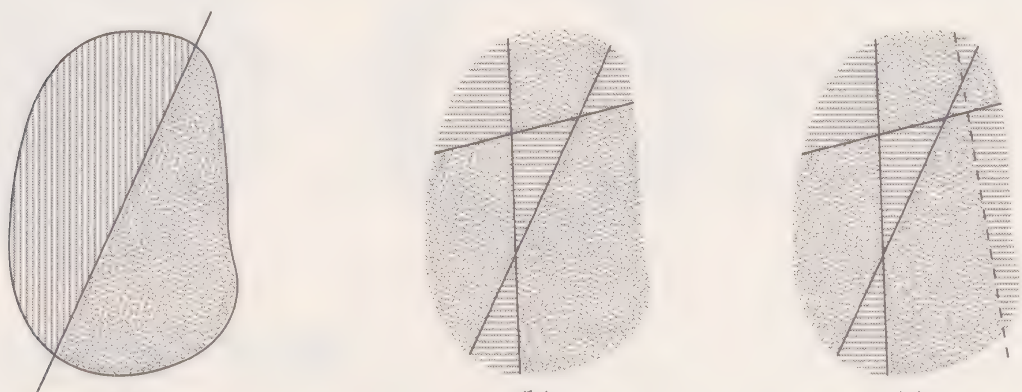
Note that in both cases one of the four colours (which one?) could also be used to colour the external face surrounding the map. Of course, the problem of colouring does not depend on the isomorphic representation that could be made of a graph.

Graphs that can be coloured with two or three colours

What form must maps (polygon graphs) have so that they can be coloured with two colours? And with three colours? These questions do not present us with any great difficulties and can be explained relatively briefly. First the theory for two colours, which states the following:

> A map can be coloured with two colours if and only if all the vertices of its equivalent graph have an even degree of greater than or equal to two.

Strangely, if the map is coloured with two colours the vertices of its graph have an even degree because if there were a vertex with an odd degree, at least one face would be adjacent to at least two more faces and, therefore, three colours are needed. To establish the reciprocal, various steps must be considered. Firstly if n straight lines are 'thrown' randomly onto a plane, it can be demonstrated that two colours are sufficient for colouring in the resulting linear map (think of a chess board!). To do so we will use the method of proof by induction, which consists of proving the first case $n = 1$ and seeing that if something works for the value of n it will also work for a value of $n + 1$.

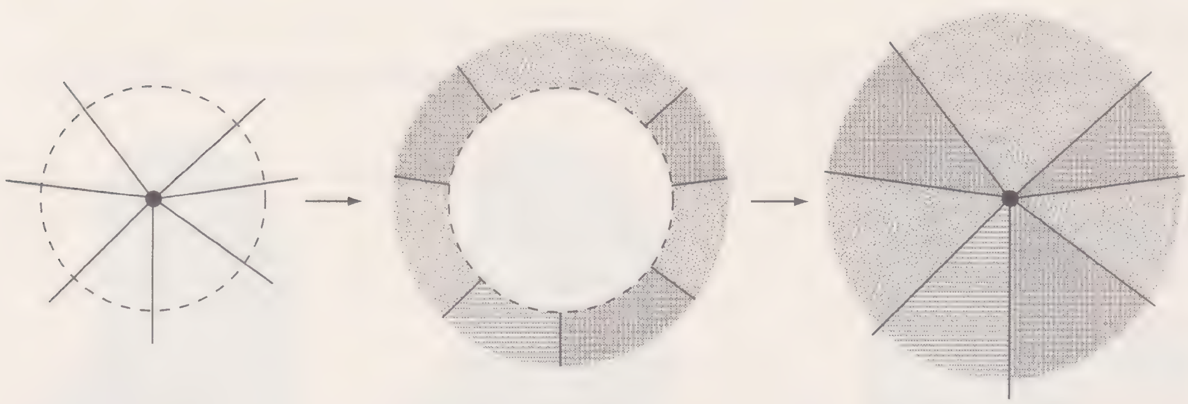


(a) (b) (c)

For $n = 1$, a straight line (a), the result is simple. Let's assume that the result works for n straight lines (b) and focus on the map for $n + 1$ straight lines (c).

Removing one of the straight lines leaves a map with n straight lines, which can be coloured with two colours (by inference) and so, by adding straight line $n + 1$, the same colours are used for the areas above (or to the right) of this additional straight line and they are swapped on the other side. And so the case of $n + 1$ is solved with just 2 colours. The reader can confirm *mutatis mutandi*, that all maps defined by n circles distributed randomly on a plane can also be coloured with two colours. Note that, both in the case of straight lines and circles, the resulting graph has vertices of even degrees. For any graph with vertices of even degrees greater than two, by eliminating a cycle or a chain, the resulting graph continues to have vertices of even degrees and, as with all graphs of this type, it will allow isomorphic representation formed by straight lines and circles. The result for two colours is now clear.

Regarding colours, on many occasions we are only interested in considering graphs with a degree of no more than 3 on each vertex: If there were one vertex V with a degree of greater than 3, drawing a circle C with centre V that does not cut any other vertex, by eliminating the lines inside the circle, we will again get a new graph with vertices with degrees of 3 corresponding to the intersections of C with the original edges. Therefore, if this map is coloured in, by removing the circle and returning to the original graph, the problem is resolved, as shown in the figures below. Thus, for purposes of colouring, sometimes we can restrict ourselves to considering polygon graphs where every vertex has a degree of 3.

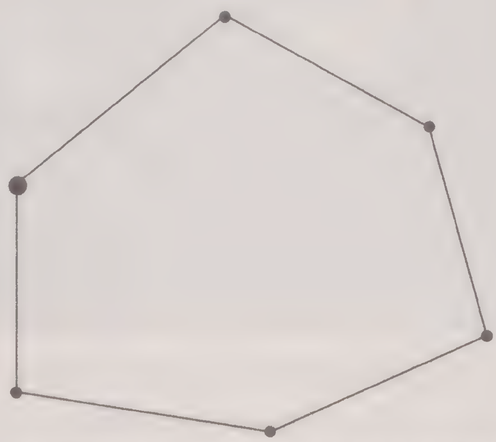


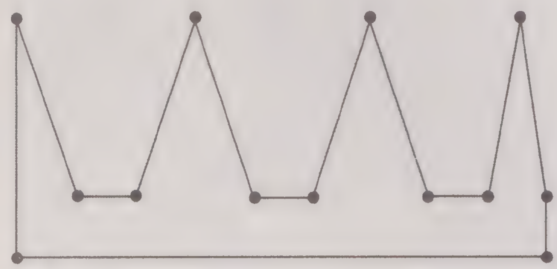
The theorem of three colours requires a more complex justification, which is omitted here, states the following:

> A polygon graph (with vertices with degrees of 3) is colourable with three colours if and only if each face is limited by an even number of edges.

MUSEUM GUARDS AND COLOURING GRAPHS

The problem of distributing the guards among the rooms of a museum inspired Victor Klee to come up with the following problem in 1973: For a polygon-shaped museum with one floor and *n* walls, if we want to locate guards so that they can see all the walls without moving, how many guards do we need? In the first figure we can see a convex polygon which is very easy to monitor with a single guard in one corner, but in a concave polygon, such as the following figure, many more are needed. The result is that if there are *n* walls it is sufficient to strategically locate [*n* / 3] (where [ ] indicates an integer part of the division, in other words, divide and eliminate decimals).



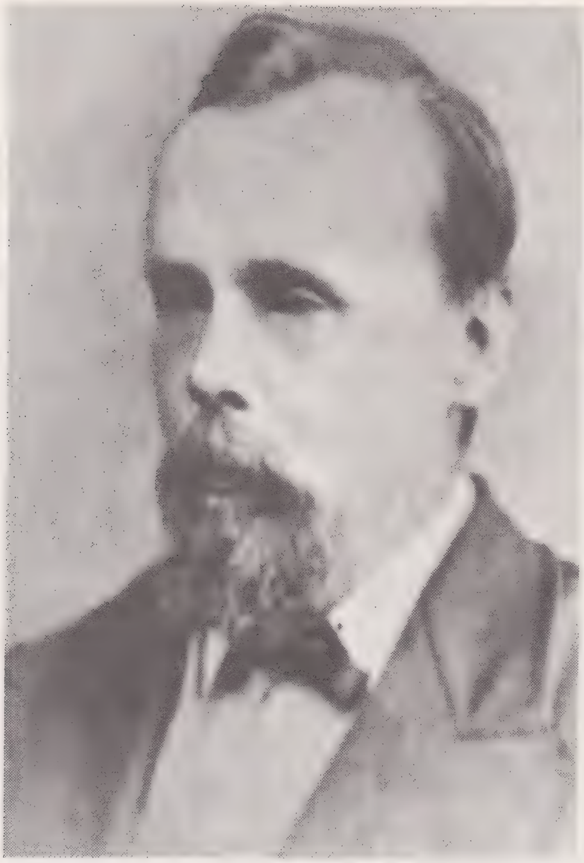
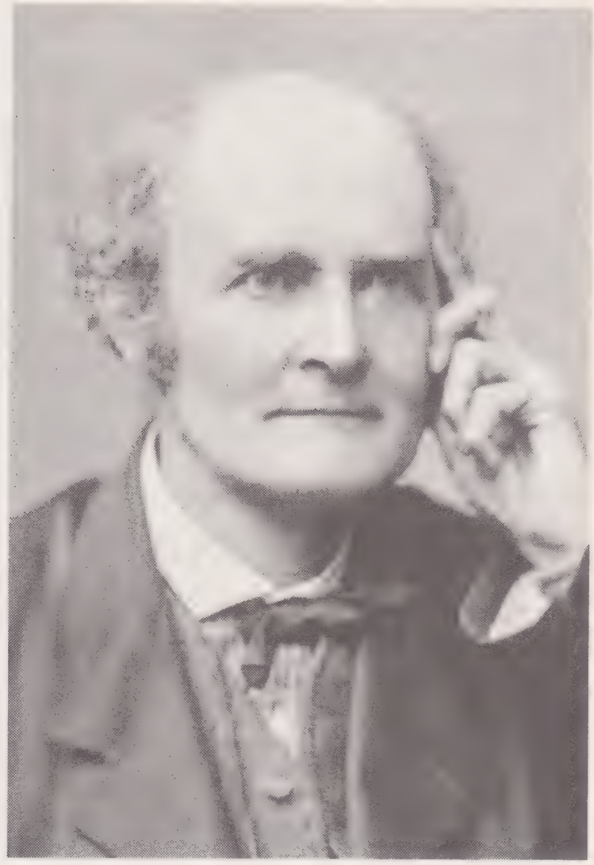


The strange thing about this issue is that the previous result is justified as the graph formed by the triangularisation of the room associated to the vertices (drawing the convenient diagonals between them) is a graph the vertices of which can be coloured in with three colours in such a way that adjacent vertices have different colours.

Once the graphs that can be coloured with 2 or 3 colours were known and having soon seen that 5 colours were sufficient, it was a great challenge to solve the problem of 4 colours. It often occurs that the most specific case is the most difficult.

Four colours are enough

In 1852 Francis Guthrie had the intuition that it was possible to colour them with four colours so that the areas with common borders had different colours. As Francis had already finished his university studies, he sent a note on this interesting issue to his brother Frederick, who was at the time taking a course in mathematics with the

Mathematicians Francis Guthrie (left), who proposed the possibility of maps with four colours, and Arthur Cayley, who submitted the challenge to the London Mathematical Society.

well-known Augustus de Morgan. Not able to answer, De Morgan explained the problem to other colleagues, such as Sir William Hamilton. In 1878 Arthur Cayley formally presented the challenge to the London Mathematical Society, and thus the problem was open to everyone's consideration.

In 1879 an article was published demonstrating that, in effect, four colours were enough for all maps. The ingenious explanation is owed to Arthur B. Kempe, who was a lawyer in London. Between 1879 and 1890, Kempe's idea was agreed to be sound and therefore the problem of four colours was considered solved.

The surprise came in 1890 when P.J. Heawood demonstrated a non-resolvable fault in Kempe's solution, meaning that the problem once again needed a proof. Heawood himself and others then dedicated many years and a lot of effort to trying to prove the truth of the result. No one could ever find a map that needed five or more colours... Therefore, it was logical to expect that four colours were always enough. As always happens in mathematics, the advantage of attacking an unresolved problem leads to the development of many other results which, although they do not solve the initial problem, are useful in other fields.

DENES KÖNIG (1884–1944)

Hungarian mathematician Denes König studied in Budapest and Göttingen. It was at the latter establishment where he was fascinated by Minkowski's conferences on the problem of the four colours. König decided to dedicate his life of research and teaching to the theory of graphs, writing a treatise on graphs in 1936, which helped greatly to popularise the subject all over the world. He was unable to resolve the problem of four colours... but he resolved many others.

Interestingly, for maps located on surfaces that are more unusual than a plane or a sphere, the minimum number of colours could be demonstrated. Thus, for example, seven colours are needed on a torus (which is shaped like a rubber ring) and a Möbius strip (which is made by joining the two ends of a long rectangle after having turn one end over) requires six. A correct demonstration was also given of the fact that five colours were sufficient and the maps that could be coloured with two or three colours were defined.

In 1950 it was shown that maps with less than 36 countries could be coloured with four hues. German Heinrich Heesch, following on from Kempe's ideas, suspected that the new power of computers could help to attack this problem in which the fact of considering any map led to defining thousands of possibilities.

Between 1970 and 1976 mathematicians Kenneth Appel and Wolfgang Haken of the University of Illinois in Urbana-Champaign managed, using a computer to define thousands of cases, delivered the good news: "Four colours are enough." The event was so momentous that the United States postal service even designed a stamp with the phrase on it.

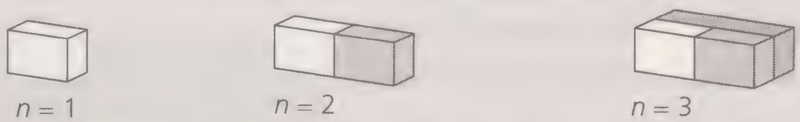
Although there have been more refined presentations of Appel and Haken's result, until now no one has managed to escape the intricacy of their proof. In other words, no

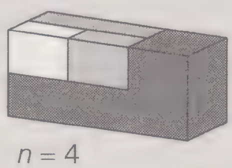
Kenneth Appel and Wolfgang Haken in a photograph taken in the 1970s.

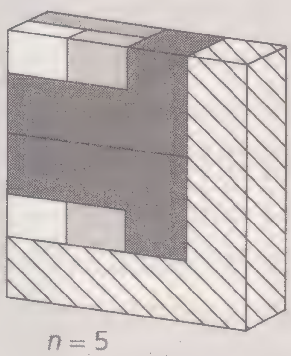
one has managed to find an argument that resolves the question without the help of a computer. This inclusion of computer sciences in mathematical demonstrations (beyond colours) has opened a new paradigm in the world of classic mathematical proofs. In 1997 Robertson, Sanders, Seymour and Thomas managed to update this demonstration using both old ideas and new computer resources. However, a 'classical' demonstration is still pending.

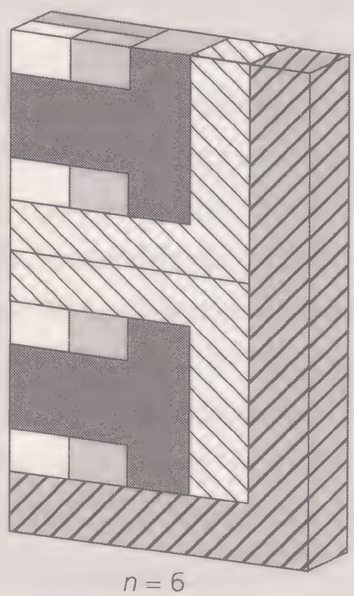
Subsequently, new colouring problems have arisen. For example, Herbert Taylor has suggested generalising the problem of four colours in the following way: how many colours are necessary to colour a map on which each country or area to be coloured is formed by m unconnected parts, assuming that all the territories in the same country have to be the same colour and the regions of the same colour do not share a common border. When $m = 1$ we return to the problem of four colours. Heawood demonstrated in 1980 that for $m = 2$, 12 colours are needed. For $m = 3$, H. Taylor has demonstrated the need for 18 colours and for $m = 4$, 24 colours are required. For $m \geq 5$ there is a conjecture that states that *6m* colours will be enough but this problem is still open to question. Other diverse problems on colouring maps today constitute a doctrinal *corpus* on the theory of graphs that continues to attract the interest of researchers.

INFINITE COLOURS IN SPACE









If, instead of maps on a plane, we consider bodies in space that we want to colour in such a way that bodies with a common face have a different colour, the result is extraordinarily surprising. Four or six colours are not sufficient, instead we need... infinite colours, as can be seen in this sequence drawn by C. Alsina and R.B. Nelsen in 2006.

A PROBLEM PROPOSED BY PAUL ERDÖS

What is the minimum number of colours needed to colour in a map so that any pair of different points a unit apart are in regions of a different colour? Leo Moser verified that four are certainly needed… is that enough?

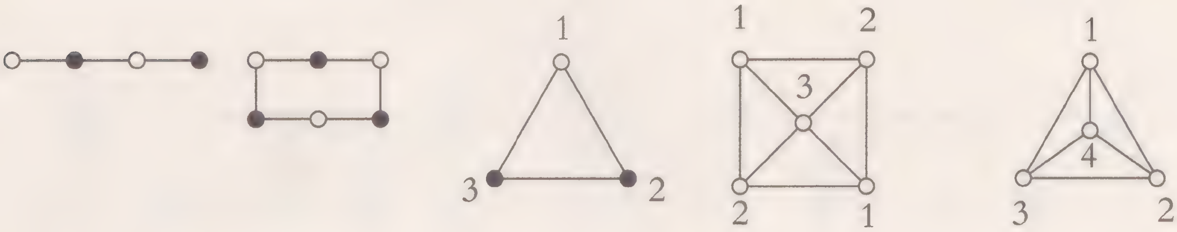
The chromatic number

In the same way that the colouring of faces of graphs has been considered, we can also consider the colouring of edges and vertices.

Colouring vertices $V(G)$ of graph G, given a group of colours C, consists of assigning each vertex a colour C so that all vertices connected by one edge are given a different colour. In this context, the *chromatic number* of the graph G is defined as $X(G)$, the minimum number of colours necessary to colour G following the restriction that adjacent vertices have different colours.

If G has at least one edge $X(G)$ must be greater than or equal to 2 and, evidently, $X(G)$ cannot exceed the number of vertices V (it would be an extreme case where each vertex was painted in a different colour). Of course, the chromatic number is non-variable as completely equivalent graphs (isomorphs) have the same chromatic number.

Now look at the following graphs:



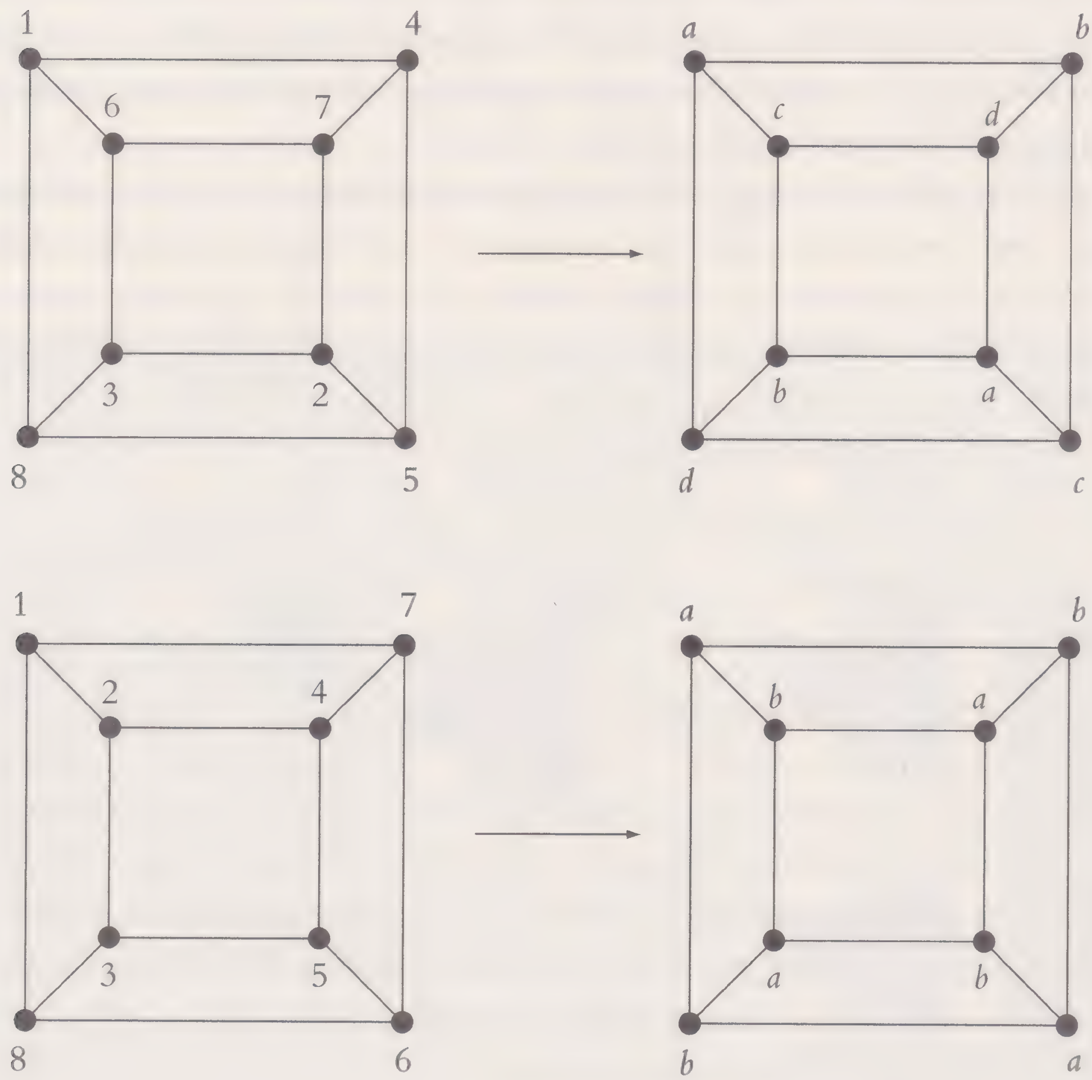
In the case of the n vertices in a line the chromatic number will be 2; alternating the colours is sufficient. This is also true of any tree. In the case of a cycle we can see that if there is an even number of vertices the chromatic number is 2, but if there is an odd number of vertices the chromatic number must be 3. Finally, in the case of wheels the chromatic number is 3 if, in the exterior cycle there is an even number of vertices and 4 if there is an odd number.

Through the concept of 'duality' we can move from one type of graph to another and use the same colour themes on faces and chromatic themes for vertices.

The interesting thing is that in place of countries, we can use linguistic categories or attributes and different groups of vertices with the same colour or category to form a classification. This occurs in the drafting of lists.

To colour the vertices of graphs, we can use the greedy algorithm, which consists of ordering the vertices by numbering them, assigning a first colour to the first on the list, then giving the second a colour (if it is adjacent to the first the colour should be changed and if not, it is repeated), and so on. But great care is needed, because this algorithm does not necessarily carry the same chromatic number and, therefore, a a final check will be needed to minimise the number of colours.

The following figures show how a graph equivalent to a cube can be improved from four colours to a beautiful solution with just two by applying the algorithm.



In fact, the only thing the greedy algorithm can assure in terms of colouring is that as a rule, the number of them will be the maximum of the degrees of the vertices plus one. Finding efficient colouring algorithms is not a trivial problem after all.

This chapter has allowed us to see something that appears frequently in mathematics – a challenge that starts as a game but becomes enormously productive.

COLOURS, GRAPHS AND POEMS

The English poet J.A. Lindon, surprised by so many people's fondness for trying to prove that four colours are enough to colour the graphs, wrote the following ditty:

Hues
Are what mathematicians use
(While hungry patches gobble 'em)
For the four-colour problem.

The mathematicians may have taken their time but now, rather than the four-colour problem, we have the four-colour theorem.

Chapter 3

Graphs, Circuits and Optimisation

I believe that this nation should commit itself to achieving the goal, before this decade is out, of landing a man on the Moon and returning him safely to the Earth.
John F. Kennedy, 25 May 1961

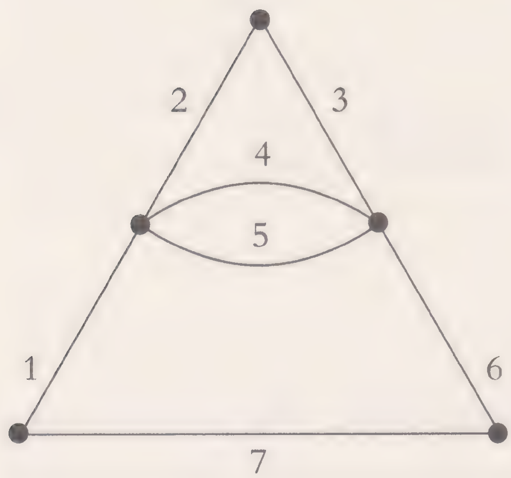
Oh God, now we really have to do it.
Robert F. Freitag (NASA)

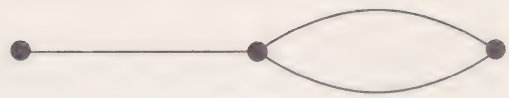
In the second half of the 20th century, graph theory, on the cusp of making a spectacular mathematical development, acquired a new dimension by forming part of many applications related to planning and optimising systems. Technological advances and the flourishing field of computer sciences were added to this development, but never had the idea of achieving optimum solutions, in time or cost, led to such a search for efficient methods and algorithms. NASA's grand programme for the launch of Apollo 11, collecting rubbish and cleaning in large cities, supply chains and car and food distribution chains, all needed convenient methods for proposing good solutions. The operative research was highly successful and graph theory aroused great interest that lives on today. This chapter invites you to appreciate the potential of this theory in practical problems.

Eularian circuits

Using a connected graph to find a Eularian circuit is to find if it is possible to start from one vertex of the graph and return to it passing along all the edges of the graph just once (the vertices may be repeated, but not the edges).

A Eularian circuit can be seen in the first of the following figures, while in the second graph it is not possible to find one.





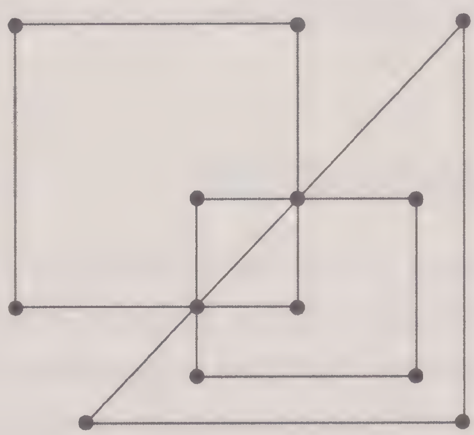
Euler managed to clarify perfectly whether a connected graph is a Eulerian circuit by considering the key concept of *degrees* (or valency) of the vertices, which describes the number of edges that join with them. The key to the problem is given by Euler's theorem, which states the following:

> "A connected graph contains a Eularian circuit if and only if all the vertices have an even degree."

It should be noted that when there are Eularian circuits, all the edges must be paired with another in the route and it is then natural that the vertices have an even number of edges, in other words, even degrees. Knowing this means that having counted the degrees of the vertices we immediately know if there is a Eularian circuit. But actually finding that circuit can prove to be a much more difficult problem.

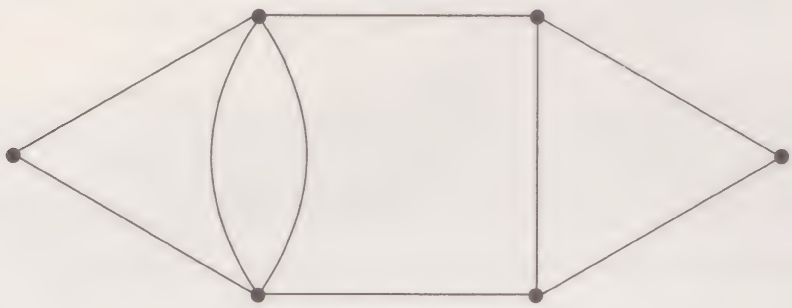
RECREATIONAL EULARIAN CIRCUITS

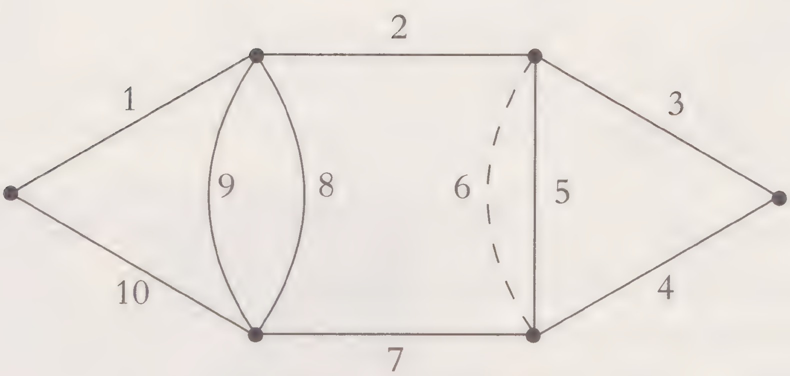
A classic mathematical game (requiring a pencil and piece of paper) is to see if it is possible to trace a line from one vertex of a graph (without removing the pencil from the paper), passing along each edge only once and return to the starting point, in other words, table-top Eularian circuits – ideal for paper napkins. Try it with this figure.



The Chinese postman problem

The ideal route for an intelligent postman who wants to do their work well (travelling along all the streets where they need to deliver letters) would be one in which each street only needs to be walked once. Drawing a graph to represent the streets is an ideal way to look for the Eularian circuit. But if the Eularian circuit does not exist, some of the streets will have to be repeated, trying to limit the number of repetitions to the minimum possible. As this problem was studied by Chinese mathematician Meigu Guan in 1962, it has become popularly known as "the Chinese postman problem".

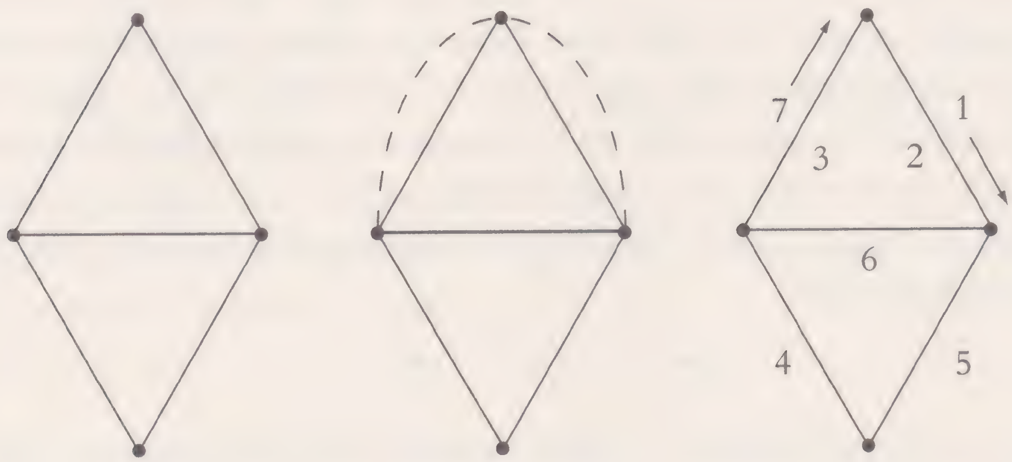




If you look closely at the above figures you will see that there are two vertices with degrees of 3. Therefore you already know it will be impossible to find a Eularian circuit. However, in the second figure it can be seen that with the trick of introducing just one new edge (represented by the dotted line) it now has a Eularian circuit, with the route indicated by numbers, in which just one street is travelled along twice (shown as 5 and 6, the added edge). This method for solving the Chinese postman problem is called *Eulerising the graph*.

Thus, so that we can say "he who Eulerises well will be a good Euleriser" we proceed as follows. If there is no Eularian circuit, add the minimum number of edges necessary, duplicating some of the existing ones, until there is a Eularian circuit.

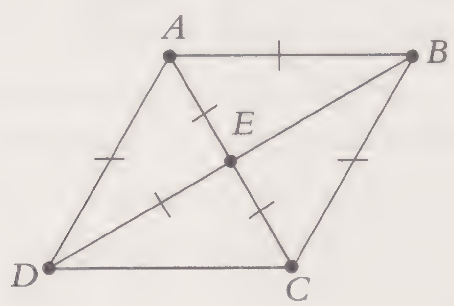
In the following figures we can see a possible Eulerisation and the postman's final route.



Other than postmen, the problem described here has applications in all types of distribution (and collection) services. In big cities, for everything from street cleaning to commercial distribution, having suitable road routes is of great significance in keeping the costs down. Fortunately, nowadays computers help in the efficient design of these routes.

Hamiltonian circuits

Consider the following problem for a connected graph. Can a path be found that starts from a vertex and travels through some of the edges, allowing all of the vertices to be passed through just once, and then returns to the start vertex? If this route is possible it is called a *Hamiltonian circuit.*



In the above figure the path *DABCED* would be Hamiltonian. Although they are similar, Hamiltonian circuits should not be confused with Eularian circuits. In

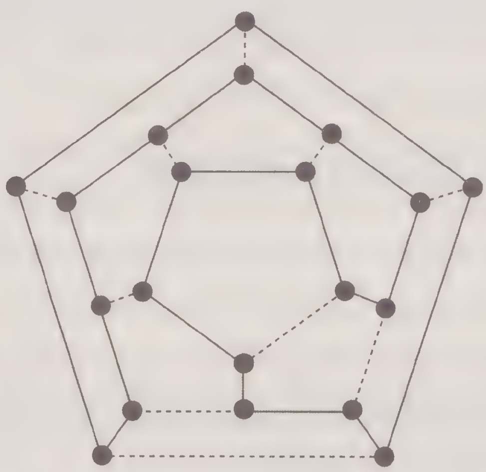
Eularian circuits you have to be able to pass along the edges only once (remember the bridges of Königsberg), while in the case of Hamitonian circuits it is the vertices that cannot be repeated.

In many cases there will be no Hamiltonian circuit, while in others there may be several. For example, in the figure, *DABCED* and *DCEBAD* are Hamiltonian. Of course if one of these circuits exists, the same route in the opposite direction would also be valid.

Despite the difficulty presented by large graphs when it comes to determining Hamiltonian circuits, the problem is of great interest for the organisation of excursions, all types of collection, the distribution of goods in supermarket chains, etc.

A TWO-GUINEA INVENTION

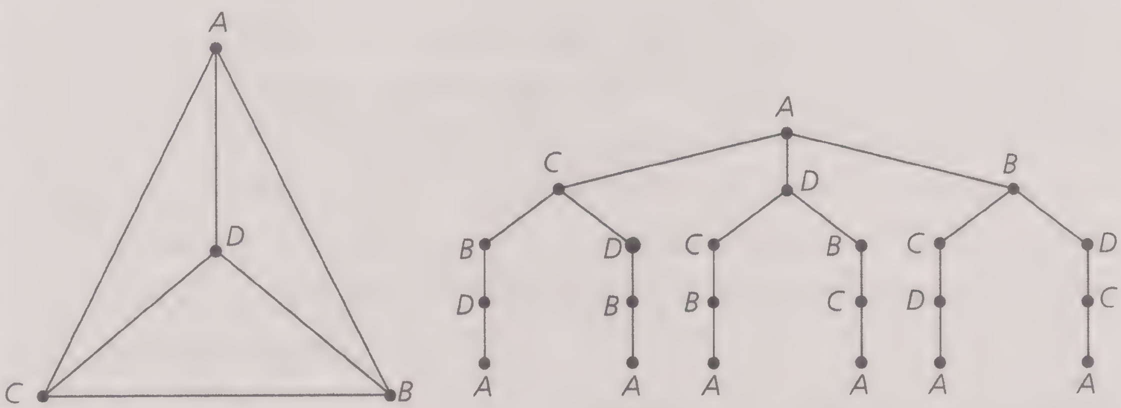
Although the initial idea for circuits in graphs came from Thomas Kirkman (1806–1895), the mathematician who popularised and researched them was Irishman William Rowan Hamilton (1805–1865). In 1859 Hamilton took a dodecahedron (which is a regular polygon with 20 vertices), placed 20 names of cities at the vertices and came up with the game of finding a route on the dodecahedron, which started at any city and returned to the same one having visited the other 19 cities just once. Excited by his own game, he sold the idea to a games manufacturer which gave him the paltry sum of two guineas. Good ideas do not always receive the economic recognition they deserve – although the game was not a commercial success.



Mathematician William Rowan Hamilton and his dodecahedron game.

THE TREE METHOD

In the figures in this box we can see how a complete tree of possible routes can be associated to an initial graph *ABCD* in order to find Hamiltonian circuits which, starting from *A*, return to *A* having passed through *B*, *C* and *D* just once. Numbering Hamiltonian circuits is complicated; so, in each case we have a complete graph with n vertices for each city (and there are n) each one leads to $n-1$ and each one of them to $n-2$..., until returning to the start. Consequently, in each case we have $(n-1)\ (n-2)\ (n-3)\ldots\ 3\cdot 2\cdot 1$ routes. Remembering that the factorial of a natural number is the product of that number and all previous numbers down to 1 ($6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$) there are $(n-1)!$ circuits. But as we can always draw these circuits in the reverse direction, the true figure is half that number $(n-1)!/2$.... Nevertheless the numbers are enormous. With just $n=6$ it would already be $(6-1)!/2=60$ routes.



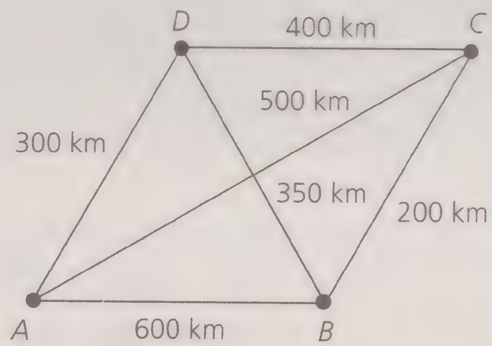
The traveller problem

The previous section covered the possible Hamiltonian circuits, starting at a vertex and passing through all vertices just once and returning to the start point. In most cases we do not just have a graph, but also values associated with the edges (cost per trip, distance, etc.) and, therefore, we do not just want to find a circuit, but minimise costs, time or distance.

Think of the postman, the travelling salesman or the soft drinks distributor; all of them want to do their job and return home having travelled the shortest route. When you organise your holidays this issue is also of interest to you (maybe you want to complete your journeys in the shortest time possible, or with the minimum

NEAREST NEIGHBOUR ALGORITHM

Imagine that *ABCD* are cities, the edges are kilometre distances and you start at *A*. You have three alternatives; 300 km, 500 km or 600 km; chose the nearest, *D*. From there you have the choice of 350 km or 400 km; choose the nearest, *B*. From *B* you have to go to *C* and then return to *A*. This is a type of so-called 'greedy algorithm' as, step by step, the most avaricious option is chosen (cheapest, least time, least distance, etc). This algorithm is a way of organising the route, but it does not guarantee that it always gives the optimum solution. An alternative (which is not optimum either) is the 'classified edge algorithm', in which all the edges that are added when determining the circuit are chosen in order of increased weight, which precludes circuits that impede a Hamiltonian circuit.



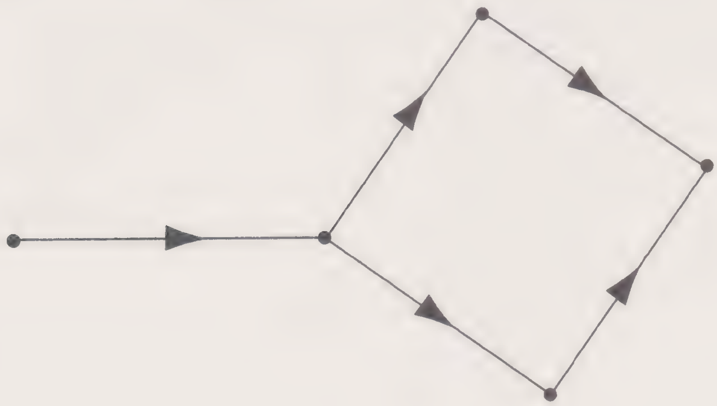
possible cost even though the trips are longer, etc.). In Chapter 5 it will be shown that this subject is of key importance in linear programming. The complexity of resolving the traveller problem for large graphs makes it an emblematic case of the so-called NP-complete problems, in other words, it is considered that it will never be possible to find a 'quick' algorithm to obtain optimum solutions. Computer science has considered the concept of algorithmic 'speed' in relation to the execution of computer programs and the time necessary for their execution.

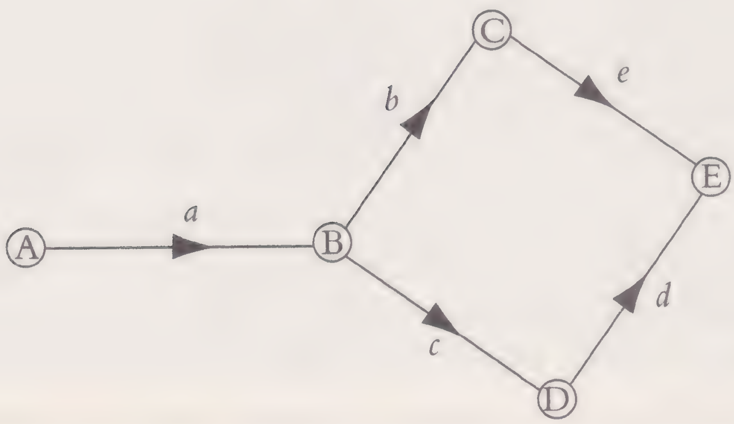
KRUSKAL'S ALGORITHM

Joseph Bernard Kruskal (1928–2010), a combinatory analyst at the Bell Laboratories who trained at the University of Princeton, came up with a remarkable algorithm in the 1950s: adding edges ordered by cost generates a minimum cost tree.

Critical paths

In many real situations it is worth going from graphs to *diagraphs* or *directed graphs*, adding arrows to the edges to indicate a specific direction or sequence. In the first figure below, the diagram could represent trips on streets in one direction. In the second figure the (same) diagraph could be representing a series of tasks (*A, B, C, D, E*) which need to be carried out in a certain order.





Electrical networks, traffic, telephone connections, industrial manufacturing plans, repair operations – they all work with diagraphs. As can be seen in the second figure, not points but circles or rectangles are often placed at nodes *A, B, C, D, E* , within which the tasks are described (unload, paint, organise, etc.) and even weights or biases for the tasks (£1,000, 12 minutes, etc.) and on the orientated edges or arcs weights valuing the cost, time, etc. demanded by that route or sequence of actions also appear.

In such cases, which are generally complex, it is relevant to find the critical paths – those that due to their cost or the time required will be decisive in order to reach the end. In the example of the top figure if times *a, b, e* add up to 34 days and *a, c, d*

OPTIMISING AIR TIME

At airports it is of particular interest to optimise the time between the arrival and departure of a plane. Once the plane lands it has a number of tasks, such as the following:

A. Unloading passengers
B. Unloading goods and luggage
C. Cleaning inside
D. Stocking up on food and drink
E. Inspecting the plane
F. Refuelling
G. Loading new goods
H. Loading new passengers

Some of these activities can be done simultaneously (for example, A and B, C and D, E and F) and others in sequence (C cannot start if A has not finished, G goes after B, etc.). The final activity is H (having finished F even if G is still in progress). If all of this has to be done in 20 minutes, a diagraph should show the critical path that will affect the planning of flights. And also the waiting time and delays!

add up to 45 days, the critical path is *ABDE*; if it has not been possible to complete the entire critical path, even though other operations have been finished, the final project will not be able to be completed.

Graphs and planning: the P.E.R.T system

After World War II, an extensive range of methods aimed at optimising planning was developed. It was the launch of the Soviet Sputnik that motivated the Americans to initiate similar grand projects, from the Polaris ballistic missiles launched from submarines to the arrival of man on the Moon. And grand projects demand suitable planning methods. Such methods configure the so-called 'mesh analysis' and among the most important examples are the following:

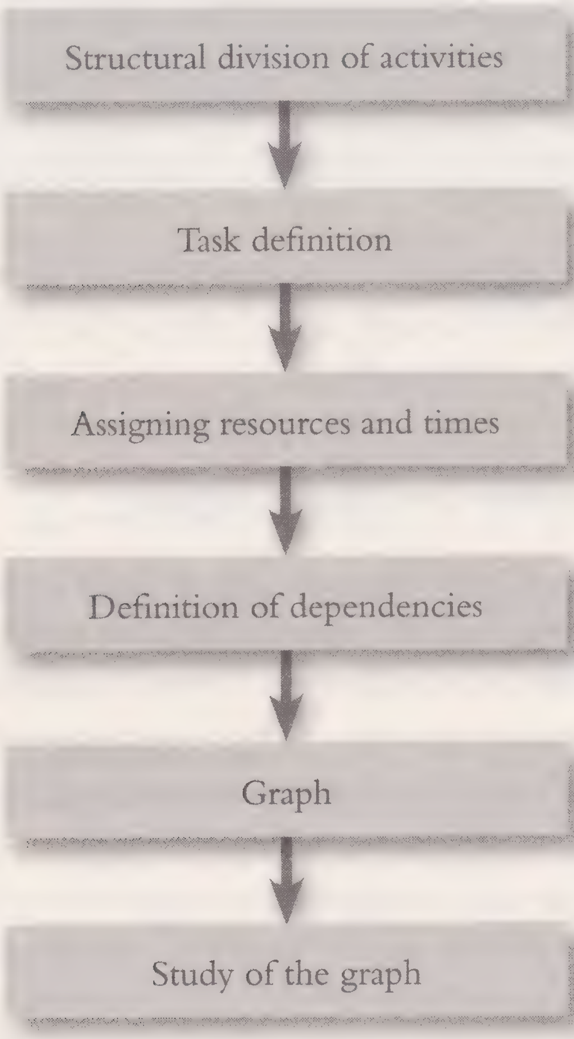
1. P.E.R.T. (Program Evaluation and Review Technique.) It was developed by the US Marines in 1958 and has been of great use in complex planning of time and costs.
2. C.P.M. (Critical Path Method.) This method was particularly applicable to temporary planning and to the study of critical paths or series of coordinated activities, which could delay the smooth progress of the plan. Other similar methods are C.P.S. (Critical Path Analysis), P.E.P. (Program Evaluation

Procedure), L.E.S.S. (Least cost Estimating and Scheduling) and S.C.A.N.S. (Scheduling and Control by Automated Networks).

3. R.A.M.P.S. (Resource Allocation and Multi-Project Scheduling.) This method expands on P.E.R.T. and is particularly used in the distribution of limited resources among several independent projects.

Organigram showing the steps of a P.E.R.T.

In general terms, a P.E.R.T. is represented by the following organigram:



PROGRAMMING TIMETABLES WITH CRITICAL PATHS

Making the best use of people and machines in a production chain is crucial in industries where products are manufactured using several machines or in a number of steps. However we can find another use for *processing algorithms*, *timetable programming* and *studying critical paths*, establishing the dependencies or independence of the tasks of central importance. Ronald Graham created a processing algorithm for a list of activities with m processors evaluating that if T is the optimum time, the algorithm will not exceed $(2 - (1/m))T$. Here a processor is a person, machine or system with programmed times. In the *decreasing time algorithm*, in which the longest activities are first, the time does not exceed $[4/3 - 1/(3m)]T$. But specific *non-relational solutions* should never depreciate.

Generally, the P.E.R.T. system that we are dealing with here is based on the following principles.

1. *An organised structural distribution of the project activities is established.* This distribution can be done by means of an organigram to help visualise in detail the essential activities that make up the project, as well as the groups of activities that have to be completed by each of the teams involved in the effective progress of the project.

2. *The tasks are defined.* This specification of the basic tasks that needs to be carried out (as well as the technology they require) allows bottlenecks to be limited. This stage will specify all of the project's activities and events.

3. *Resources are assigned to each task and execution times are set.* At this point a project 'calendar' should be created specifying the global time and the partial time for the activities, taking into account all the factors that may influence the temporary planning (resources, technologies, equipment, etc.). One of the original features of P.E.R.T. is the introduction of various concepts of time:
   a) T_o: *optimistic time* or the time that would result from exceptional progress in the tasks;
   b) T_p: *pessimistic time* or the estimated time of activities that are adverse or catastrophic for the progress of the project;

c) T_m: *mean time* or probable, or time statistically evaluated from previous experiences;

d) T_e: *expected time*, which is that which is actually included in the P.E.R.T. for each activity and is calculated using the formula (of statistical justification):

$$T_e = \frac{T_o + T_p + 4T_m}{6},$$

In other words the expected time is a weighted average of the optimistic, pessimistic and mean times. A standard deviation is applied to this calculation $\frac{T_p + T_o}{6}$ the square of which evaluates the variability.

4. *Dependencies are studied and defined.* In this step all the possible dependencies between the project's activities are established (resource limitations, physical space, equipment, intrinsic order of placement, etc.).

5. *The network or graph that constitutes the fundamental model for the application of the system is drafted.* In order to create the graph, the following conventions should be adopted:

   a) The *events* (start or end of an activity) are the vertices of the graph. They are represented by circles and rectangles within which the event is written in words and its number with respect to the other events.

   b) The *activities* correspond with the edges of the graph (whether directed or not) and are established between the related events. A number is specified on each edge or activity that corresponds to the expected time T_e for that activity.

Some rules for the effective completion of the graph or network help to make the graph easier to read:

   a) Each activity has a preceding event and a final event. Fictitious activities may be introduced (time 0) which result from rewriting the same event several times (with different numbers) if different activities start from it.

   b) An event is considered complete only when all the activities that precede it have also been completed.

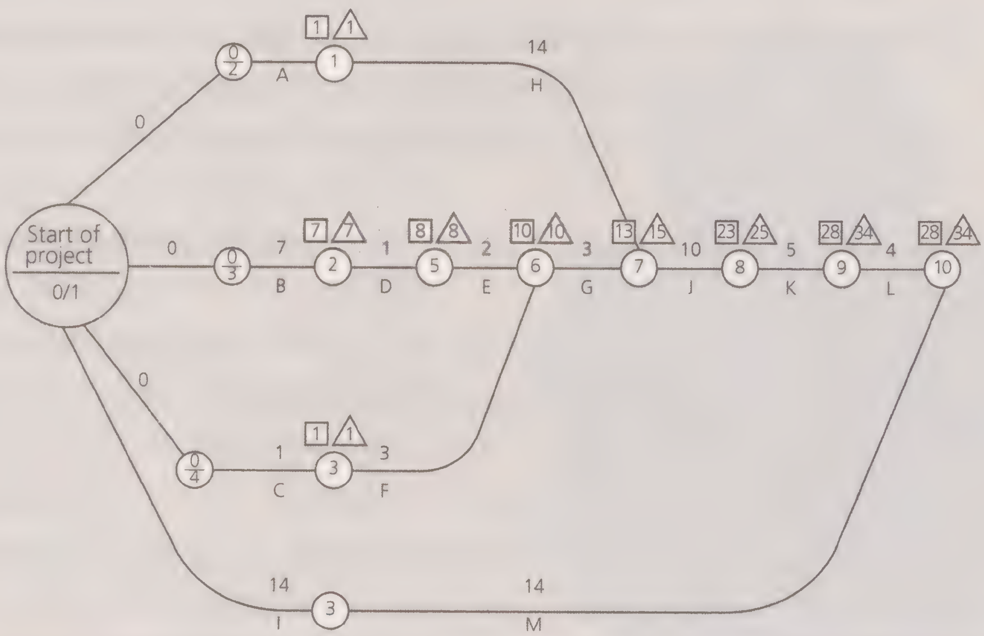
   c) Parallel activities between two events should be avoided, breaking the

EXAMPLE OF A P.E.R.T. IN CONSTRUCTION

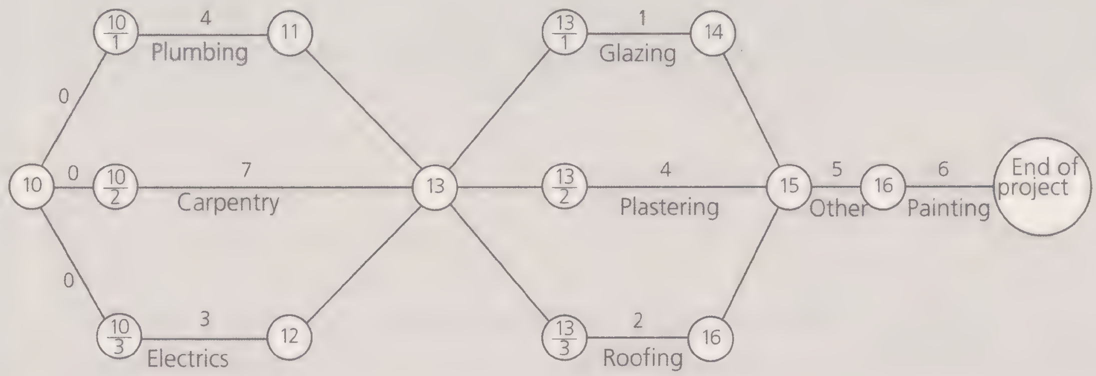
Below is a typical P.E.R.T. analysing the construction of a house, only covering the initial activities. We start by making a table of initial tasks, and a letter or number is assigned to each one, as well as an estimated time (T_e) and the determination of any dependencies:

Letter	Activity	Event start No.	Event end No.	T_e
A	Order bricks	0 (2)	1	1
B	Order equipment	0 (3)	2	7
C	Order concrete	0 (4)	3	1
D	Ground levelling	2	5	1
E	Excavation	5	6	2
F	Concrete delivery	3	6	3
G	Lay foundations	6	7	3
H	Brick delivery	1	7	14
I	Woodwork design and order	0	4	14
J	Construction of walls	7	8	10
K	Construction of framework	8	9	5
L	Construction of covers	9	10	4
M	Delivery of woodwork	4	10	14

Now the corresponding graph can be drawn placing a square and a triangle outside each vertex. Inside the square the day on which the event could be started is specified, and in the triangle, the day on which it should finish.



A typical expansion of this graph to the conclusion of work, would be the following:



parallelism with fictitious activities (time 0).

d) Intermediate fictitious events and activities must be created to eliminate vertices of the graph of 4 degrees or greater.

e) No event can be both initial and final on a path of activities on the network.

6. Finally, the P.E.R.T. graph is analysed. For example, the following parameters are of interest:

   a) *Maximum advanced date* of the end or beginning of an event following a path of activities.

   b) *Final allowable date*, in other words, the last date at the end of an event which can be allowed without altering the general organisation.

   c) *Slack of an event*, which is the difference between the two previous times.

   d) *Margin of an activity* or the excess time that can be used in carrying out an activity

   e) *Critical path*, the path of a graph that requires the longest time to carry out (from two given events or the entire graph).

The so-called P.E.R.T./COSTS system is carried out with the same premises, but taking into account the costs of the activities instead of the time taken to carry them out. Mixed P.E.R.T.s can also be made of cost and time.

Chapter 4

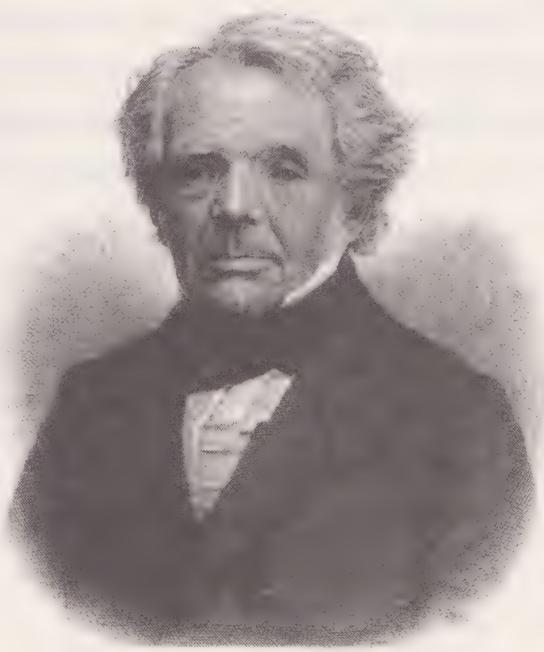
Graphs and Geometry

Inspiration is needed in geometry,
just as much as in poetry.
Alexander Pushkin

Much of geometry depends on measuring objects: angles, distances, perpendiculars, surface areas, volumes, etc. However, the considerations of graph theory and topology have helped to clarify geometric facts that do not depend so much on measurements but relate to the configurations of points and lines. This short chapter invites you to enjoy Euler's famous formula and use it to derive some surprising results from the geometry of polyhedrons and mosaics.

Descartes' 1640 formula and Euler's from 1752 are based only on faces, vertices and edges but were still applicable to many different shapes and continued to be valid when the shape was deformed. This would give rise to a new branch of mathematics called *topology,* which was developed in the 19th century. A.F. Möbius, B. Riemann, H. Poincaré, L.E.J. Brouwer, S. Lefschetz and many other mathematicians from diverse specialities found this 'new geometry' to be one of the foundations for studying everything from curves, areas and spaces to functions. They used topology to make discoveries that would not have been possible to formalise in the traditional framework of geometry.

In brief, it could be said that topology is free from the rigid structures of

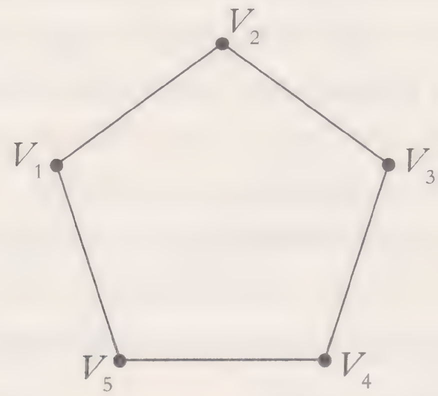


Portrait of August Ferdinand Möbius, one of the 19th-century mathematicians interested in topology.

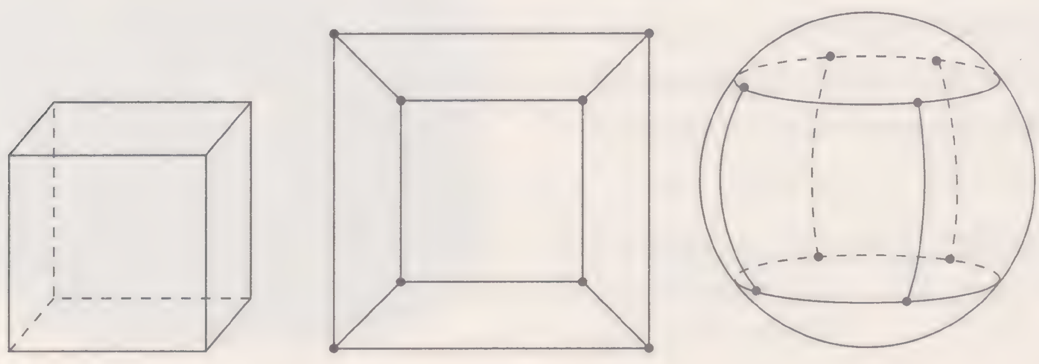
Euclidean geometry, or any projective geometry. By allowing 'continuous deformations' it manages to model a new world of shapes and use new categories of transformations. Imagine a triangle drawn on the surface of a balloon. If the balloon is squeezed (without popping it) the poor triangle takes on different shapes in which angles and lengths will vary, although the 'triangular essence' of the figure determined by three points and three lines between them will be maintained. Thinking about figures as deformable rubber is a good visual resource for thinking topologically. So, a sphere would never be deformed into a doughnut, but a doughnut (with a hole) could be shaped into something as seemingly different as a coffee mug.

Euler's surprising formula

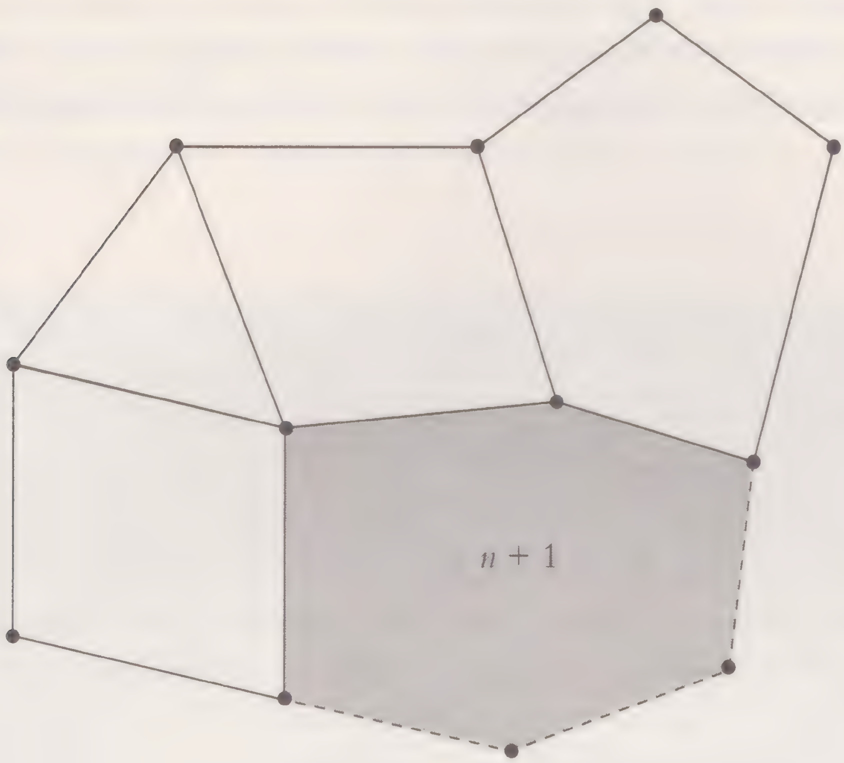
Consider a convex polygon with its n vertices V_1, V_2..., V_n and the corresponding edges V_1V_2, V_2V_3..., $V_{n-1}V_n$, V_nV_1.



Besides the lengths of the sides, the angles, the straightness of the edges, etc., a ratio that always applies is that the number of edges is equal to the number of vertices, a ratio that is so trivial that it could go unnoticed. If the vertices are maintained and a simple curve replaces a straight edge between them, the vertex/edge ratio will be maintained.



Now let's consider a space and any convex polygon described by V vertices, E edges and F polygonal faces. If the polyhedron is projected from a point inside onto a large sphere, the corresponding lines and vertices will be marked on the sphere, such that the values of V, E and F will be maintained in the spherical configuration.



The polyhedron can also be made to correspond with a polygonal map which has the same number of edges E, the same number of vertices V, and F faces.

So it can therefore be seen that if $F = 2$ we have a polygon and $V = E$, or similarly, $F + V = E + 2$. If for $F = n$ it has V_n vertices, E_n edges then it follows that $n + V_n = E_n + 2$, so for $F = n + 1$ we should focus on just one face (the $n + 1$ face). This configuration is obtained by adding a certain amount of K vertices and $K + 1$ to a map with n faces, V_n vertices and E_n edges, therefore,

$$F+V_{n+1} = n+1+V_n+K = (n+V_n)+(K+1) = (E_n+2)+(K+1) =$$
$$= (E_n+K+1)+2 = E_{n+1}+2$$

And so Euler's famous formula is established, which states the following:

The ratio $F + V = E + 2$ applies to all convex polyhedrons.

If you think about it a little bit, although it may seem fairly ordinary at first, this is a surprising relationship, as it applies to any convex polyhedron – the type of faces, the angles of the polygonal faces, the angles between the planes of the faces and the lengths of the edges are all unimportant. A formula that applies to infinite and greatly varied figures should catch the eye. It is not the norm. There are barely any formulae that apply to such a variety of shapes. It is a subversive formula that pokes fun at measurements to give a purely combinatorial numerical relationship.

GIVEN $E = F + V - 2$, CAN I DETERMINE F AND V?

If we have a convex polyhedron, *then* $F + V = E + 2$ and, therefore,

$$E = F + V - 2. \tag{1}$$

But, what reasonable values can we have for F and V? Are there any restrictions that should be taken into account for F and V? Can $F = 1{,}000$, and $V = 2$? Here you can discover the simple restrictions for F and V.

Evidently $V \geq 4$, as with fewer than 4 vertices it would not be a polyhedron, and each vertex should have at least 3 edges, so $3 \cdot V \leq 2E$, as each edge is determined by two vertices. Therefore $3V \leq 2F + 2V - 4$, which gives:

$$4 \leq V \leq 2F - 4. \tag{2}$$

Also $F \geq 4$, as at least four faces are needed to enclose a piece of space and we need at least three edges for each face, so, $3F \leq 2E = 2F + 2V - 4$, from which:

$$4 \leq F \leq 2V - 4. \tag{3}$$

Relationships (1), (2) and (3) correspond to the convex polyhedron in space. The most simple examples of polyhedrons with any number of faces $F \geq 4$ are pyramids and bi-pyramids: With a polygonal base of $2K$ edges and an exterior point we get a pyramid with $F = 2K + 1$ and two of these pyramids joined at the base gives a bi-pyramid with $F = 4K$.

From Euler's formula on convex polyhedrons associated to a sphere we can consider the so-called 'Euler-Poincaré characteristic':

$$\chi = V - E + F.$$

In the case of a sphere, we have seen that $\chi = 2$. If we consider a torus (a circular surface generated by a circle that rotates around an exterior axis), then we get $\chi = 0$ and, therefore, for "toroidal polyhedrons" it is $0 = F + V - E$. The genus of a surface

$$g = \frac{1}{2}(2 - \chi)$$

corresponds to the number of 'holes' in it (in the sphere $g = 0$, and, therefore, for toroidal polyhedrons it is $g = 1$...). Thus, the χ or the g are described as 'characteristics of the surface', in other words, the '2' in $F + V = E + 2$ hides the presence of the 'spherical characteristic' of convex polyhedrons. This relationship of Euler's is not valid for concave polyhedrons.

Staying in the world of convex polyhedrons, the following sections will explore the consequences of $F + V = E + 2$ in greater depth.

Euler's formula with just faces and vertices

You now know the limits for the number of faces F and for the number of vertices V of a convex polyhedron. The number of edges E completely depends on F and V. What we are going to propose below is to eliminate E from Euler's formula.

The price that should be paid to completely eliminate the number E is simply to be 'more explicit' with F and V, specifying what hides behind those numbers.

Starting with a convex polyhedron P with F faces and V vertices indicated as F_n the number of faces with n – edges and with V_n the number of vertices with n – edges. So you can write the following (finite) sums:

$$F = F_3 + F_4 + F_5 + F_6 + \ldots \qquad (1)$$

And also:

$$V = V_3 + V_4 + V_5 + V_6 + \ldots \qquad (2)$$

As an edge belongs to two faces at the same time it must have:

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + \ldots = 2E \qquad (3)$$

And as each edge joins two vertices, it will also be:

$$3V_3 + 4V_4 + 5V_5 + 6V_6 + \ldots = 2E \qquad (4)$$

If we then bring in Euler's formula doubled, $2F + 2V = 4 + 2E$, using (1), (2) and (3) will give:

$$2F_3 + 2F_4 + 2F_5 + 2F_6 + \ldots + 2V_3 + 2V_4 + 2V_5 + 2V_6 + \ldots =$$
$$= 4 + 3F_3 + 4F_4 + 5F_5 + 6F_6 + \ldots,$$

That is to say:

$$2V_3 + 2V_4 + 2V_5 + 2V_6 + \ldots = 4 + 3F_3 + 4F_4 + 5F_5 + 6F_6 + \ldots \qquad (5)$$

And similarly, using (1), (2) and (4) gives:

$$2F_3 + 2F_4 + 2F_5 + 2F_6 + \ldots + 2V_3 + 2V_4 + 2V_5 + 2V_6 + \ldots =$$
$$= 4 + 3V_3 + 4V_4 + 5V_5 + 6V_6 + \ldots$$

That is to say:

$$2F_3 + 2F_4 + 2F_5 + 2F_6 + \ldots = 4 + 3V_3 + 4V_4 + 5V_5 + 6V_6 + \ldots \qquad (6)$$

Although you may currently be a little frustrated with such cumbersome equations, you should be glad at having translated Euler's formula into explicit relationships between faces and vertices, without edges.

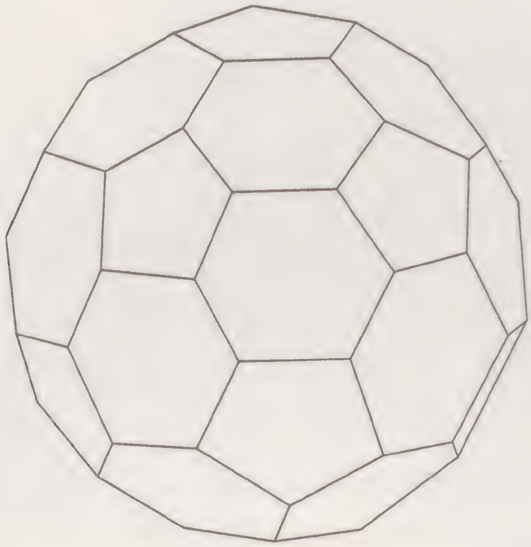
If you add (5) and the double of (6) you get:

$$2V_3 + 2V_4 + 2V_5 + 2V_6 + \ldots + 4F_3 + 4F_4 + 4F_5 + 4F_6 + \ldots =$$
$$= 12 + F_3 + 2F_4 + 3F_5 + 4F_6 + \ldots + 2V_3 + 4V_4 + 6V_5 + 8V_6 + \ldots$$

Simplifying this gives us a wonderful expression:

$$3F_3 + 2F_4 + F_5 = 12 + 2V_4 + 4V_5 + ... + F_7 + 2F_8 + ... (\star)$$

Where the edges are not explicit, the hexagonal faces do not appear and nor do the vertices with three edges. Enjoy ($\star$) and memorise it: it will provide enormously interesting discoveries. To start with, have you ever taken a look at a football? It is a semi-regular polyhedron that combines pentagons and hexagons and in which each vertex receives three edges.



Are there any other polyhedrons with these types of faces and vertices? Note that $F_3 = F_4 = F_n = 0$ if $n \geq 7$, $V_4 = V_n = 0$, $n \geq 5$..., therefore, according to ($\star$) it must be $F_5 = 12$ but F_6 is not determined (B. Grünbaum and T.S. Motzkin have proved that in fact F_6 can take any value other than 1). A curious dozen of pentagonal faces.

Regarding quadrilateral and hexagonal combinations ($\star$) will give that $2F_4 = 12 + 2V_4 + 5V_5 + ...$, in other words, it will have at least 6 quadrilaterals (and if the vertices are of 3 degrees, exactly six quadrilaterals). If triangles and hexagons are combined it must be that $3F_3 = 12 + 2V_4 + 4V_5 + ...$ and it will have at least 4 triangles (and if the vertices are of 3 degrees, exactly 4 triangular faces).

There is always a triangle, a quadrilateral or a pentagon

Let's think a little about imaginary polyhedra. Can you think of a convex polyhedron that has neither a triangle, a quadrilateral nor a pentagon? Of course not, there is no such convex polyhedron.

Go back to the (*) formula from the previous section:

$$3F_3 + 2F_4 + F_5 = 12 + 2V_4 + 4V_5 + \ldots + F_7 + 2F_8 + \ldots \quad (\star)$$

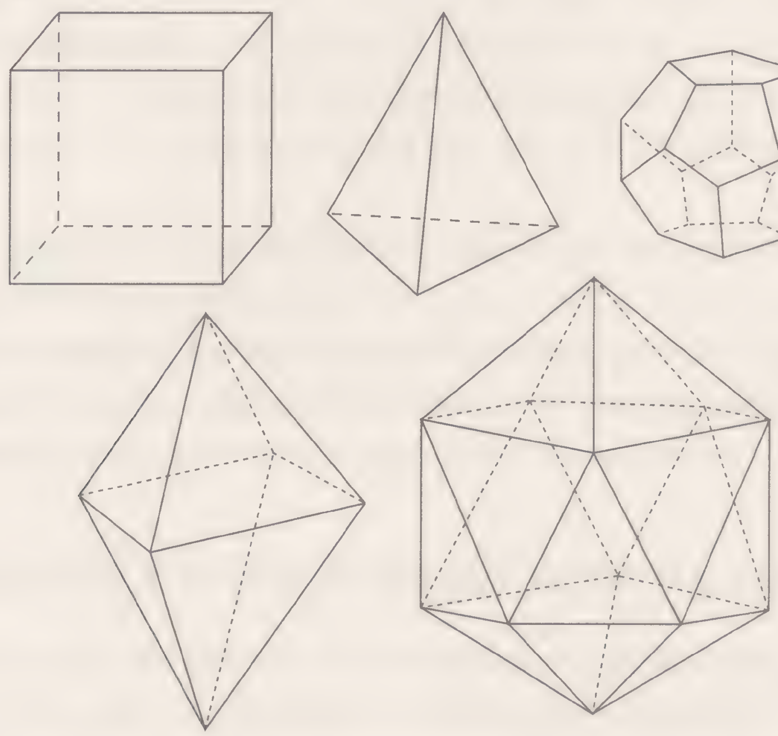
Note that the right-hand side of the formula will have at least a 12, so we always find that

$$3F_3 + 2F_4 + F_5 \geq 12,$$

then the numbers F_3, F_4 and F_5 cannot be simultaneously null... The following theorem can now be established:

> In all convex polyhedrons there is always at least one triangle, one quadrilateral or one pentagon.

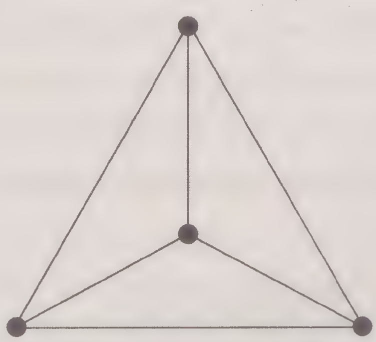
There may be other faces, but at least one face with 3, 4 or 5 edges must exist. Remember that a regular polyhedron is a convex polyhedron where all the regular polygonal faces are identical and all its vertices receive the same number of edges, That helps to explain the following formula: the only regular polyhedrons are the tetrahedron, the octahedron, the icosahedron, the cube and the dodecahedron.



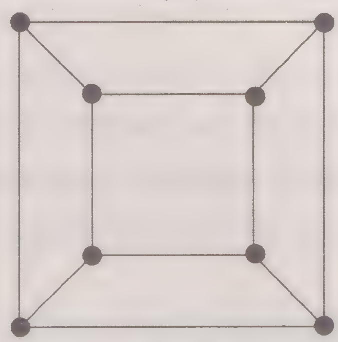
GRAPHS FOR REGULAR POLYHEDRONS

The alternative to drawing the five types of regular polyhedrons in perspective is to draw their corresponding graphs. The following figures contain the table of values of V, E and F which are shown below.

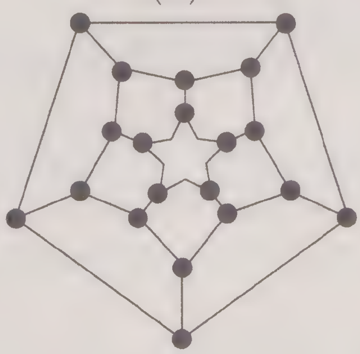
(T)



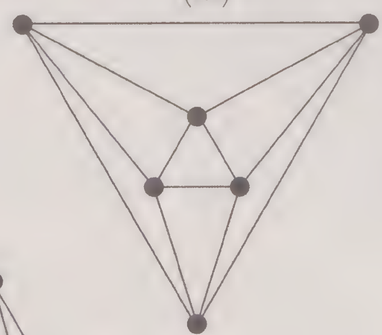
(C)



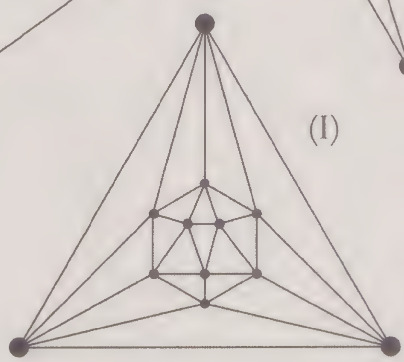
(D)



(O)



(I)



V	E	F			
4	6	4	....	Tetrahedron	(T)
8	12	6	....	Cube	(C)
20	30	12	....	Dodecahedron	(D)
6	12	8	....	Octahedron	(O)
12	30	20	....	Icosahedron	(I)

Note that this theorem combines the general Eulerian relationship with the angular characteristics of the polygons, which delimit the possible spatial corners that can be formed with triangles, squares or pentagons.

Effectively, from what you have just seen (there is always a triangle or a quadrilateral or a pentagon) and due to the definition of a regular polyhedron, the only regular ones are formed entirely by equilateral triangles or by squares or by regular pentagons.

If we only have equilateral triangles to combine, remembering their angles are 60°, the formula (*) gives $3F_3 = 12 + 2V_4 + 4V_5$. The tetrahedron has $F_3 = 4$ (and, of course, $V_3 = 4$, $V_4 = V_5 = 0$). The octahedron corresponds to the case $V_4 = 6$, $V_3 = = V_5 = 0$ and $F_3 = 8$. The icosahedron has $F_3 = 20$ and $V_5 = 12$. Squares can only have vertices with three edges, so $V_4 = V_5 = 0$ and from (*) $2F_4 = 12$, or $F_4 = 6$, the cube. Regular pentagons can only form vertices of 3 degrees, therefore (*) gives $F_5 = 12$, which is the dodecahedron.

COUNTING PROPERLY

If P is a convex polyhedron with $r(P)$ faces, consider the two parameters shown below:

$r(P)$: is the quantity of natural numbers i for which in P there is one face with i edges.
$K(P)$: is the number of sides of the face, which has the most vertices or edges in P.

Thus, for a cube P would have $r(P) = 1$, $K(P) = 4$, but in a right-angled pyramid P with a pentagonal base it would be $r(P) = 2$, $K(P) = 5$.
If P has a face with $K(P)$ sides, as each of these sides is an edge with another face, in total it would have at least $K(P) + 1$ faces, or,

$$F(P) \geq K(P) + 1.$$

As $r(P)$ itself could never exceed the cardinal number of the group {3 , 4 , 5,. ., K(P)} will be:

$$r(P) \leq K(P) - 2.$$

Thus, the previous inequalities for $F(P)$ and $r(P)$ give:

$$F(P) - r(P) \geq K(P) + 1 - (K(P) - 2) = 3.$$

If all the faces of a polyhedron were different we would get $F(P) = r(P) + 3$, which is impossible.

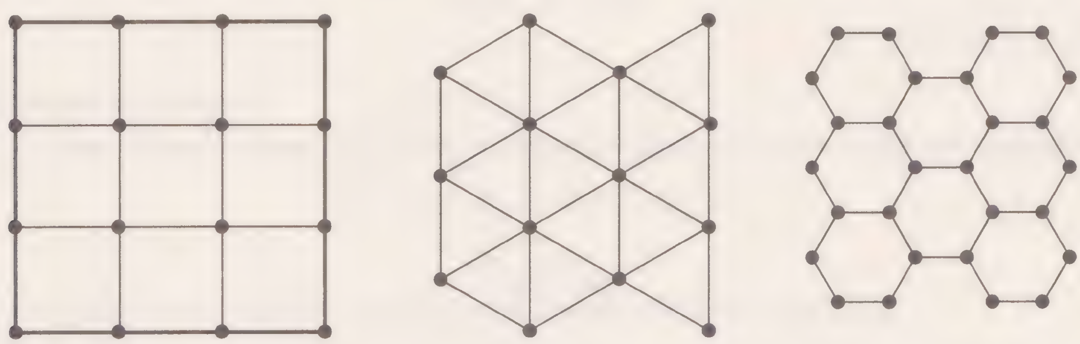
All the faces different? Impossible!

You may have now started looking for figures that do not show such repetitions. For example, you could think about how to form a convex polyhedron in which all the faces are different polygons (a triangle, a quadrilateral, a pentagon...). It would be like having the perfect sample polyhedron to take around the world and demonstrate polygons. The surprise is that this polyhedron cannot exist. And the argument against it is a very beautiful combinatorial meditation.

Think for a moment about all the convex polyhedra that you can imagine, both regular ones and weird ones. If you visualise them, one after the other, you can start to note that there are always at least some faces that are convex polygons with the same number of sides. A spatial polygonal enclosure always seems to require at least one repetition of one type of polygon.

Graphs and mosaics

Take a look at these three types of mosaic; all of them should be familiar to us, given that they appear in a lot of places.



Respectively, they are *quadrangular*, *triangular* and *hexagonal*. Each of these mosaics is a polygonal graph in the sense previously defined. In all three cases, the number of faces can be increased indefinitely so the whole of the plane can be filled. Note that at each stage of the extension, the vertices that are in the middle have a constant number of edges, except for the outside face. If the number of vertices V appear in the successive amplifications of a mosaic are counted, and in each step the number V_e of those vertices bordering the exterior face (on the bordering circuit), we will see that the quotient $\frac{V_e}{V}$ tends towards zero as V increases.

The above observations are valid for the three mosaics being considered. Below we will demonstrate a surprising result beginning with the following definition:

A regular mosaic is a polygonal graph that can recurrently cover a plane and for which the number of edges a on each vertex and the number of edges $b \geq 3$ on each face are both constant (except for the exterior face), $\frac{V_e}{V}$ tending towards zero.

The only regular mosaics (in the sense of this definition) are the triangular, quadrangular and hexagonal ones.

Effectively, if we have a regular mosaic M, where M has V vertices, E edges and V_e vertices on its border, we get $2E < aV$ given that $a\,V$ it would correspond to assign a edges to all the vertices (even those on the border). On the other hand, if the edges on the border vertices are not taken into account, we get $aV - aV_e < 2E$ Bringing both inequalities together gives:

$$aV - aV_e < 2A < aV.$$

And dividing by $2V$:

$$\frac{a}{2} - \frac{a}{2}\frac{V_e}{V} < \frac{E}{V} < \frac{a}{2}.$$

Passing the limit, when V tends towards infinity, $\frac{V_e}{V}$ tends towards zero:

$$\lim_{V\to\infty}\frac{E}{V} = \frac{a}{2}. \qquad (\star)$$

Now let's count the number of faces F of mosaic M. $F - 1$ of these faces have b bordering edges and the infinite face has V_e edges; therefore,

$$(F - 1)b + V_e = 2E.$$

Dividing by $b\,V$ gives:

$$\frac{F-1}{V} + \frac{V_e}{Vb} = \frac{2E}{bV'}$$

and passing the limit, when V tends towards infinity, another look at (*) and at the hypotheses:

$$\lim_{V\to\infty}\frac{F}{V}=\lim_{V\to\infty}\frac{2E}{bV}=\frac{2}{b}\cdot\frac{a}{2}=\frac{a}{b}. \qquad (**)$$

As mosaic M is a polygonal graph, Euler's formula is valid and can be written in the following way:

$$\frac{F}{V}+1=\frac{E}{V}+\frac{2}{V}.$$

When passing the limit it will be:

$$\frac{a}{b}+1=\frac{a}{2};$$

in other words, the constant parameters a and b are linked by the equation:

$$2a+2b=ab,$$

which can be written:

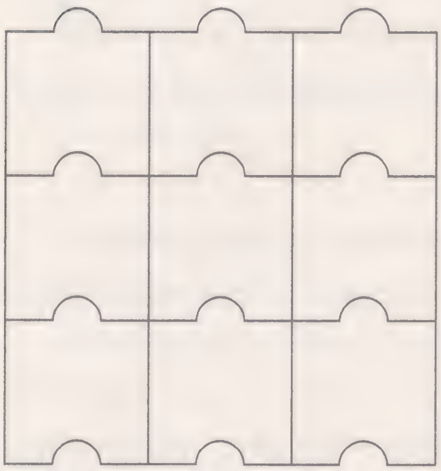
$$(a-2)(b-2)=4.$$

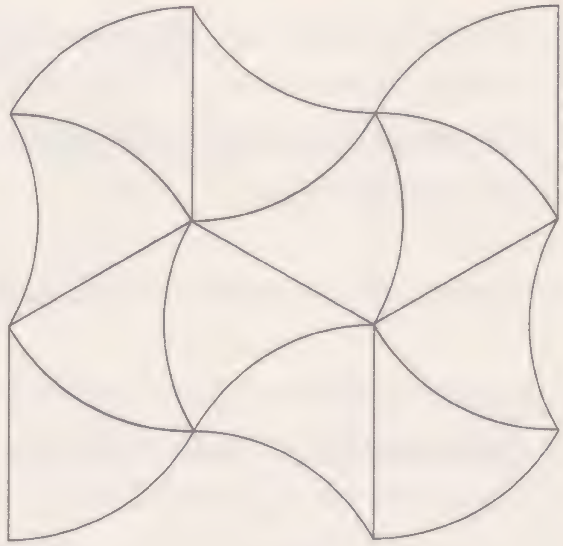
The only natural solutions are those shown in the following table:

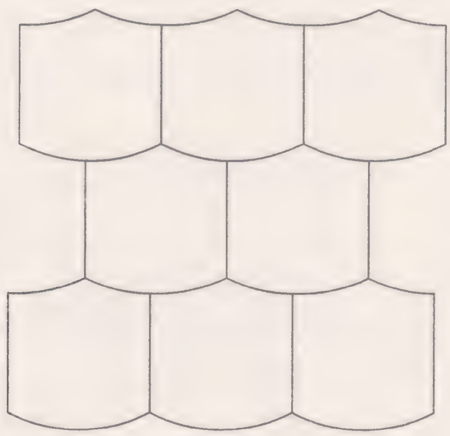
a	b		
3	6	……..	Hexagonal mosaic
4	4	……..	Quadrangular mosaic
6	3	……..	Triangular mosaic

An interesting property that should stand out is that the above demonstration, as it strictly falls within the framework of graph theory, does not depend on any other geometric property (distances, angles, parallelism, etc.) relating to the mosaic's generative figures. For example, the following mosaics correspond (except for the

isomorphism of graphs) to the three previously classified types although their different geometric appearance could make us think they are different figures.

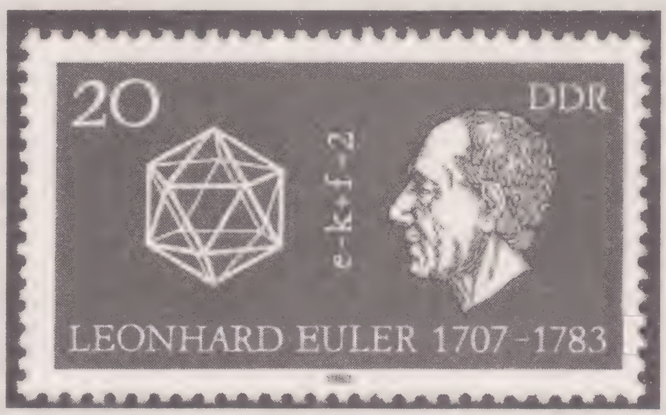






A FORMULA ON STAMPS

This stamp dedicated to Leonhard Euler was issued by the former East Germany. It has an icosahedron and the formula $E - F + V = 2$ (using the German abbreviations).



Other geometric problems with graphs

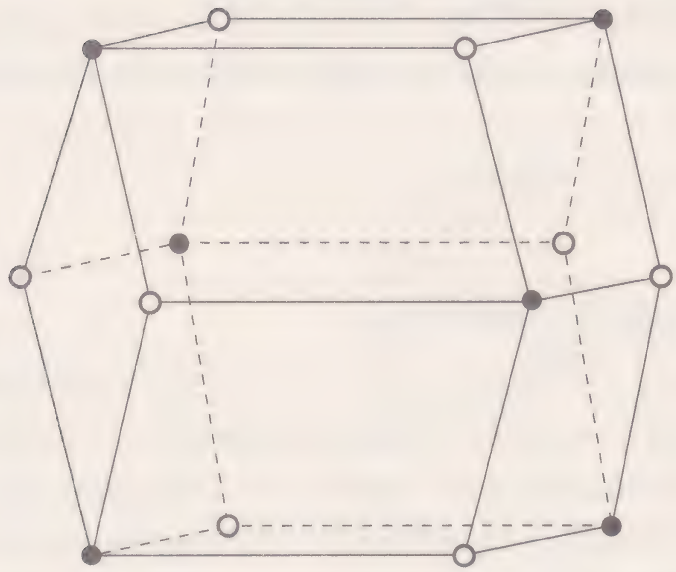
Beyond Euler's formula and all its marvellous consequences for resolving geometric problems, there are also other geometric issues that are of special interest in graph theory. Below are a few examples.

Hamiltonian circuits in polyhedrons

We have already had the chance to see how the origin of Hamilton's idea of considering the circuits that now carry his name (starting at one vertex and returning to it having passed through all the vertices just once) was a game of finding the circuit in a dodecahedron. Years later this led to the search for Hamiltonian circuits in all types of polyhedra or, where applicable, it was demonstrated that they did not exist. In the following figures, the so-called Herschel and Peterson graphs can be seen, which, for all their simplicity, do not allow Hamiltonian circuits (which readers can see for themselves by attempting to draw a circuit with a pencil).

But let's move on to three-dimensional space and, following H.S.M. Coxeter, consider the search for Hamiltonian circuits in polyhedrons other than the dodecahedron. One case very cleverly resolved by Coxeter was that of the rhombic dodecahedron.

All the faces of a rhombic dodecahedron are equal but, on the other hand, it has two different types of vertices; therefore it is not a regular polyhedron.



This interesting polyhedron represented in the diagram above has – as its enigmatic name suggests – twelve equal faces that are parallelograms, with the oddity of having 8 vertices that receive 3 edges (those marked with white circles) and another 6 that receive 4 edges (those marked with black circles). Note that the white vertices determine a cube and, therefore, you may think of the rhombic dodecahedron as a cube to which six square-based pyramids have been added. Thus, the volume of the figure is double that of the cube, and the figure, like the cube, can fill any space by repetition, like a three-dimensional mosaic.

Is there a Hamiltonian circuit in the rhombic dodecahedron? This is the problem that Coxeter answers with an absolute 'no' based on a great argument: if there were a Hamiltonian circuit, starting and ending with a vertex, we should pass through the 14 vertices just once, but each time we go from one to the next the colour should

H.S.M. COXETER (1907–2003)

Harold Scott MacDonald Coxeter was born in London and studied mathematics at Trinity College Cambridge, although his academic career began at the University of Toronto, Canada, where he worked for 60 years. He is considered one of the great geometricians of the 20th century, having written twelve influential books and a multitude of joint works with great geometry experts. He made extraordinary contributions to the study of polyhedrons and of the case of polyhedrons in spaces with more than three dimensions. Coxeter was a great friend of the famous Dutch artist M. Escher, who converted many of Coxeter's ideas into art.

be changed (from white to black or from black to white). This alternating of colours in the path is not possible as there are 6 black vertices and 8 white ones.

Graphs on non-planar surfaces

Although the graphs naturally originate from models associated with diagrams drawn on a plane, both the problems of colouring graphs and those regarding their planarity led to the study of graphs on other surfaces, such as spheres, tori, cylinders, etc. The diagrams were also placed in the third dimension, as is the case with the study of knots and their classification.

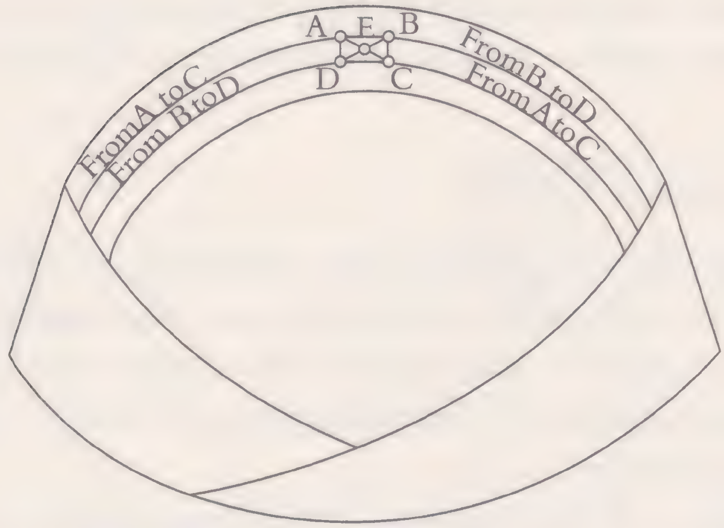
Graphs on different surfaces have helped to show many topological properties that do not vary with continuous deformations and help to classify curves and surfaces. Imagine, as we have said before, a blown up balloon on whose surface a graph is drawn with a pen. If we start to squeeze the balloon and deform it (without popping it) we will see that the graph's characteristics are maintained (number of vertices and edges, number of edges incident on each vertex, etc.).

Another example of a graph on a strange surface is a graph on a Möbius strip. If there are four points on a plane and we want to draw a graph that joins each of the points with the other three, it is not very difficult – the four points form the vertices of a quadrilateral, joining two opposite points with a diagonal and the other two by a line outside the quadrilateral, the problem is resolved. But with five points it is impossible to join each one of them to the other four without producing undesirable crossovers between the edges (the K_5 is not planar!).

The Möbius strip can be regared as a long rectangular strip of paper with the two short sides stuck together having previously turned one of the ends over. If it is not turned over and we simply stick the two parallel edges together we would get a cylinder, but thanks to its construction, the Möbius strip has the interesting oddity of having just one face. In the cylinder, the space is divided into an interior part and another exterior part, but this does not happen with the Möbius strip: there are not two faces, but one.

Is it possible to draw a graph on this surface with five points and join each point to the other four. The following diagram by Miguel de Guzmán demonstrates that what was impossible on a plane, is possible on a Möbius strip.

Miguel de Guzmán always considered games and challenges to be essential to mathematics.

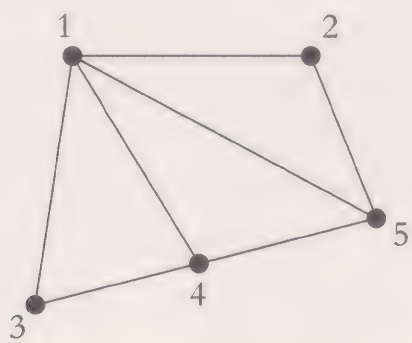


Let's draw the five points *ABCDE* on the strip, *ABCD* being a rectangle and *E* its centre joining the four vertices. Along the length of the strip (which only has one face!) the line from *B* to *D* and that from *A* to *C* can be drawn as indicated in the above diagram. Each of the five points has been connected.

Finite geometries

Imagine a plantation with various rows of trees or vegetables. Clearly, you could represent this arrangement with a graph that is formed by a series of points, without edges. But suppose we had to plan the flight of a crop-dusting plane, or the path of a fruit-picking machine. The possible 'edges' of the graph would now serve to provide possible routes for spraying or harvesting.

There are many problems that have provoked interest in *finite geometries*, in other words, the geometric systems in which there are only a finite number of points and lines that consist in certain collections of those points.



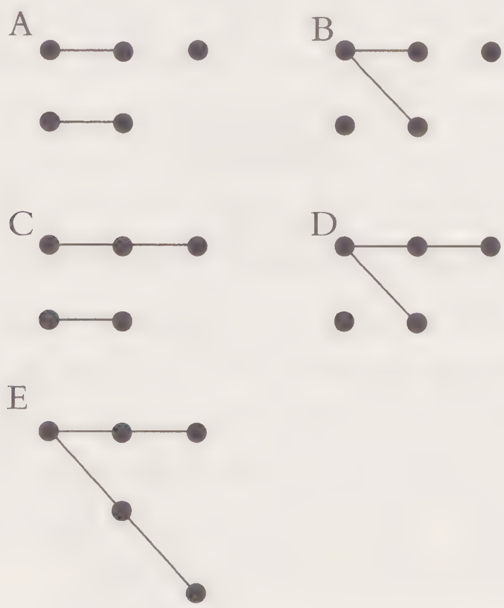
In the preceding figure, a finite geometry is represented by a graph that consists of five points 1, 2, 3, 4, 5 and the 'lines' formed by the points: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 5\}$, $\{3, 4, 5\}$. As can be seen from this example, the connection between graphs and finite geometries is an obvious one.

In the same way that traditional geometry with infinite points and straight lines can be followed using the Greek tradition started by Euclid of giving a series of axioms or properties that are taken as a starting point, several types of axioms can also be given in finite geometry and we can continue to talk of incidence (common point) or parallelism (lines without a common point), etc.

Take a look at the following example of an axiomatic system of a finite geometry:

I. There are five points and two lines.
II. Each line has at least two points.
III. Each line contains a maximum of three points.

With these rules all the possible configurations should be describable. But instead of describing the resulting sets with letters and words, it is much easier to create the possible graphs with five points and the edges belonging to them. All the possible configurations can be seen in the following figure.



In order to better appreciate the practical use of this example, think of the points as people on the board of directors of a company and the lines as committees formed

by two or three members of the board. So we can reformulate the above axioms into the language of meetings:

I. There are five people and two committees.
II. Each committee has at least two people.
III. Each committee has a maximum of three members.

This obviously gets more complicated with a lot of points and lines.

CLASSIFICATIONS AND HIERARCHIES

Just as in more traditional forms of geometry, special attention is paid to the classification of figures (triangles, quadrilaterals, etc.) and that has renewed an interest in classification problems that arise in the most diverse of subjects. The classification of shapes in computer graphics, the classification of genes, the classification of symptoms in illnesses, etc. Classification problems appear in security (digital fingerprints, iris and voice recognition, etc.) and in industrial production where quality control requires defective parts to be automatically detected and removed from the production chain.

In the case of finite sets, a relationship always comes from a set of pairs of items, and the relationships can be visualised both by means of diagrams of sets as by means of graphs, in which the vertices represent the elements and the edges of the graph join related elements. Relationships that allow classifications are called *equivalence relationships*, and they require the following properties: reflexiveness (all elements are related to each other), symmetry (if *a* is related to *b*, *b* is related to *a*) and transitiveness (if *a* is related to *b* and *b* is related to *c*, *a* will also be related to *c*). The graph associated with these relationships should reflect those properties.

Another type of relationship is that of *order*, which is used to order elements and verify the reflexive, transitive and anti-symmetric properties (if *a* is related to *b* and *b* to *a*, then $a = b$). The graphs corresponding to this order relationship in finite sets can be directed (with arcs or arrows) to indicate when an element is smaller than another or it has non-orientated edges, but then it will be agreed that the graph will be interpreted from the bottom up in order to establish order. Hierarchical processes are also of interest where priorities need to be established or certain initiatives organised (investments, construction, finding public services, etc.). In all these fields graph theory helps to understand the problem and to find its solution.

Chapter 5

Surprising Applications of Graphs

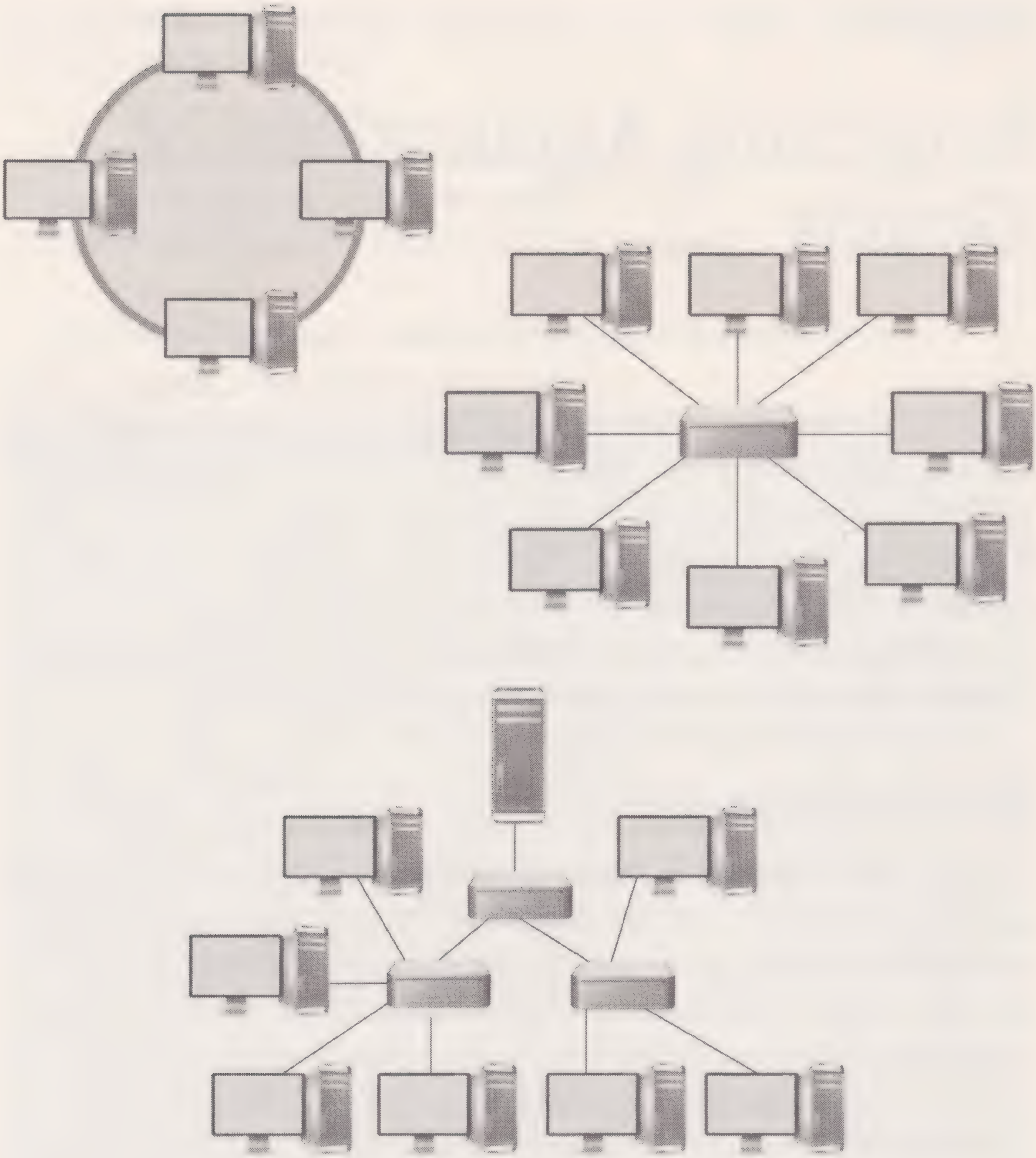
If people do not believe that mathematics is simple, it is only because they do not realise how complicated life is.
John von Neumann

You have had the chance to see the many uses of graphs in the previous chapters. This final chapter makes a note of some less obvious applications beyond the use of graphs for making maps, showing routes and connecting family trees.

Graphs and the Internet

It has often been said that the phrase Stone Age is not very apt and that it would have been more correct to talk about the Thread Age, as beyond the use of stones as tools, the decision to join these stones by means of threads was very important. In our age, the 'network of all networks', the Internet, has facilitated the digital revolution by connecting computers and servers on a global scale. Computers were becoming more powerful (while getting smaller), but what has allowed the colossal leap in the digitisation of the world has been a massive network of connections. Here, graphs and telecommunication have always gone hand in hand.

Despite the fact that human ingenuity created the first abacuses and some legendary calculating machines, no-one could have predicted that disciplines as sophisticated as cybernetics and computation would have an effect on the complex and diverse world of communication so soon. Without a doubt, it was a huge leap made in a very short space of time.



Computer networks can be connected in many ways and all of them give rise to a certain type of graph, such as the ring network (top left), the star network (top right) and the tree network (bottom).

The diagrams above show various configurations of interconnected computers. They all have an associated graph (ring, tree, star…) which is why we talk about the 'topology of networks'.

All types of connection affect the yield and functionality of the network, the physical graph of the distribution of machines, wiring, etc., other potentials, the communication protocol between machines (Ethernet, Token Ring, etc.) and the

nodes and links that establish them must all be defined. We are still at the beginning of an unpredictable process of development.

What began as a military communication project was extended to university-level research before finally becoming a global network of Internet users. Search engines were developed to access to this complex world of connections and queries, of links that guide the way. All this forms a graph with incredible dimensions (and which continues to grow).

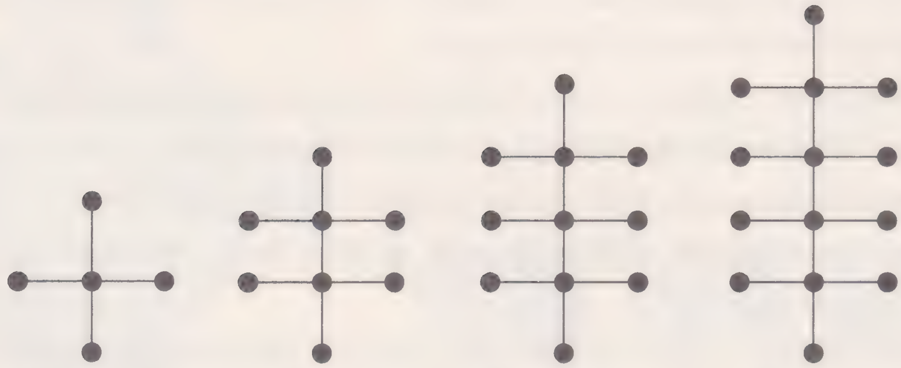
A search engine such as Google can access more information now than has ever existed in the past. But, in order to avoid total chaos, Google has employed a page tracker (the Googlebot) and has adopted complex algorithms for *ordering the list* of the items searched for. The following description from Google itself, some years ago, gives a detailed idea of how the weightings and orders which appear in searches (PageRank) are created:

> "PageRank relies on the uniquely democratic nature of the web by using its vast link structure as an indicator of an individual page's value. In essence, Google interprets a link from page A to page B as a vote, by page A, for page B. But, Google looks at more than the sheer volume of votes, or links a page receives; it also analyses the page that casts the vote. Votes cast by pages that are themselves 'important' weigh more heavily and help to make other pages 'important'.
>
> "Important, high-quality sites receive a higher PageRank, and are placed higher on the list. Thus, PageRank is Google's general indicator of importance and it does not depend on a specific search. Rather it is the characteristic of a page, based on web data that Google analyses using complex algorithms which evaluate the link structure."

Graphs in chemistry and physics

Graphs are of great interest in the representation and research of special molecular structures. The simplicity of graphs is very helpful for understanding links in molecular complexity or chemical isomers.

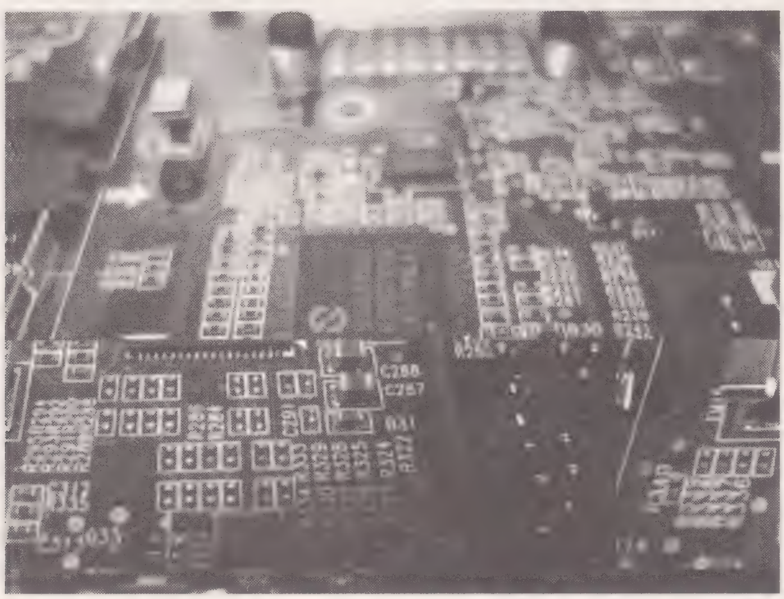
Anyone who has studied organic chemistry well knows how graphs are used to represent the different compounds.



Methane Ethane Propane Butane

Isobutane

Graphs are also used in diverse technological branches, such as electrical circuits and integrated circuits.



Graphs are also present in modern telephone circuits.

A 2,400-TONNE GRAPH

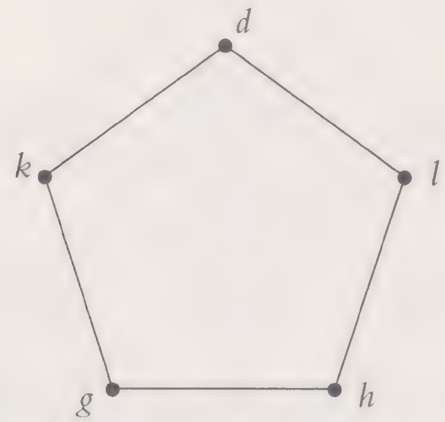
The *Atomium*, an impressive 103-metre-high steel structure, was built for Expo' 1958, held in Brussels. Its designer, engineer André Waterkeyn, was inspired by the graph that represents a molecule with 9 steel spheres (with diameters of 18 m) and 20 connection tubes.

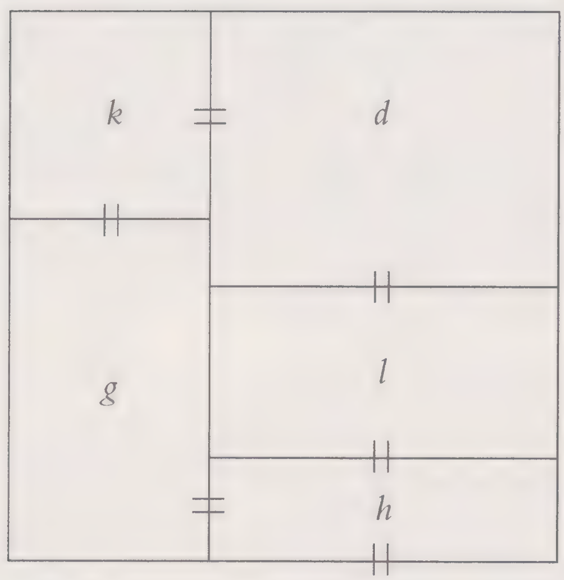
Graphs in architecture

Graph theory is a key link at the different stages of the completion of an architectural project. Once the parts and elements that will form the project are profiled, and before starting to draw the first sketches, it is helpful to draw the graph of relationships between the fixed elements. Different types of relationships of course: physical access (doors), visual access (window, glass...), common wall, etc., lead to several graphs on the same set of objects – as many graphs as there are types of relationships. Let's look at some simple examples.

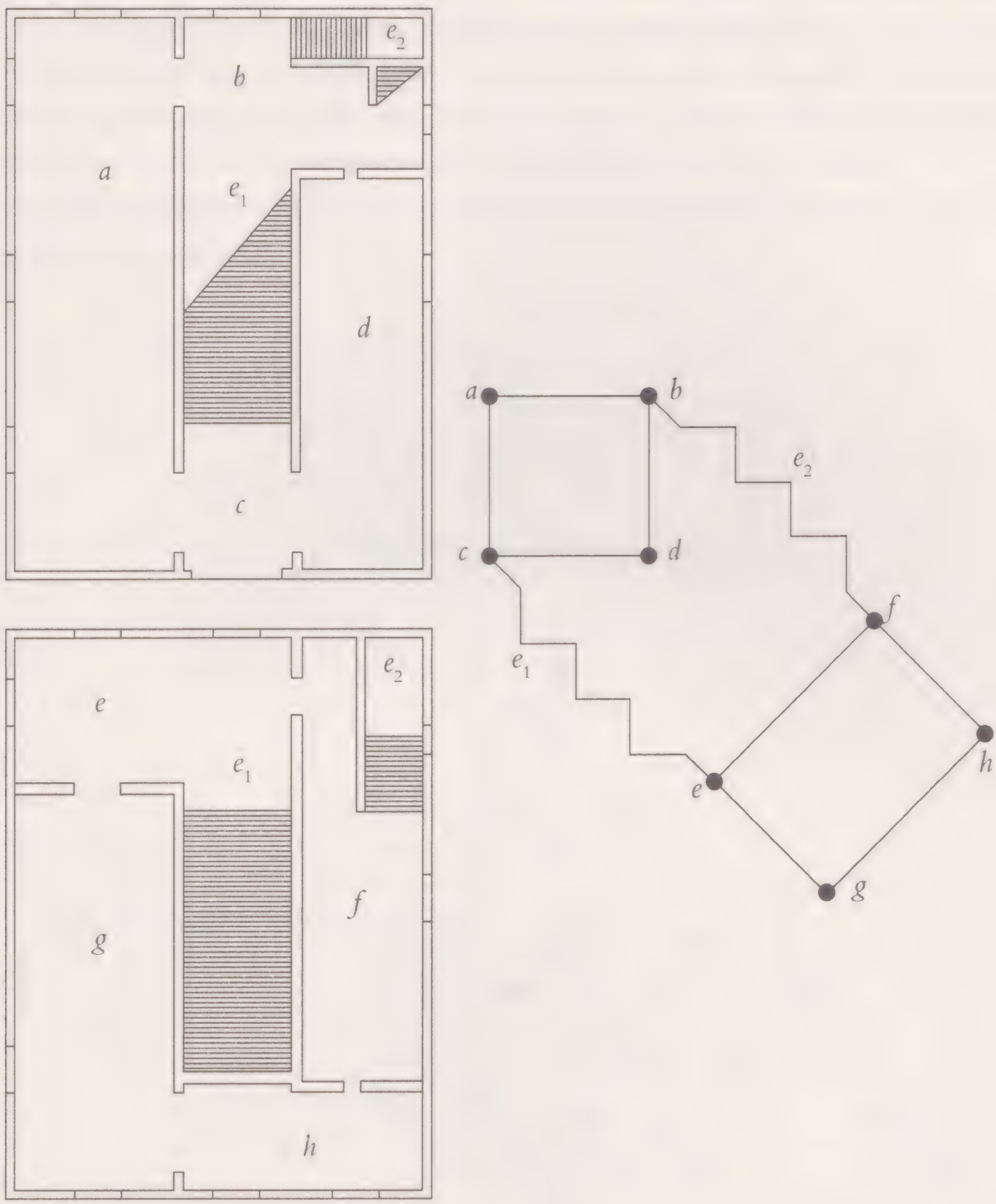
On the ground floor of a single-family home (of rectangular shape) the following elements need distributing: a kitchen (*k*), a dining room (*d*), a lounge or living room (*l*), a hallway (*h*) and a garage for a car (*g*). The following access is required: garage to kitchen, kitchen to dining room, dining room to living room, living room to hallway and hallway to garage.

If we represent the elements with points *k, d, l, h* and *g* and we draw segments between these points as edges that symbolise the relationship 'access to' we will get a graph, which represents a cycle. With this distribution it is possible to establish a circuit between any elements. The graph allows several solutions to be tested on paper.





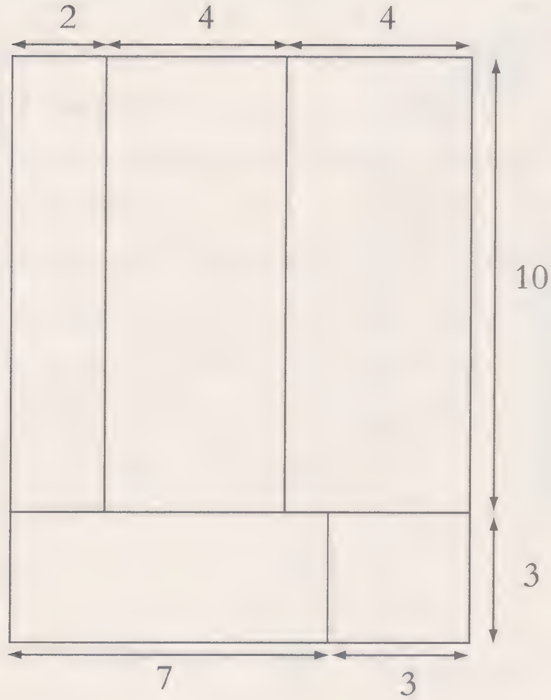
The points that indicate exterior space or communicating stairs can also be marked. In the case of multi-story housing, a graph of accesses or connections is drawn for each floor and the accessible points of the various floors are joined, not with a segment but with a zig-zag line to represent the stairs.

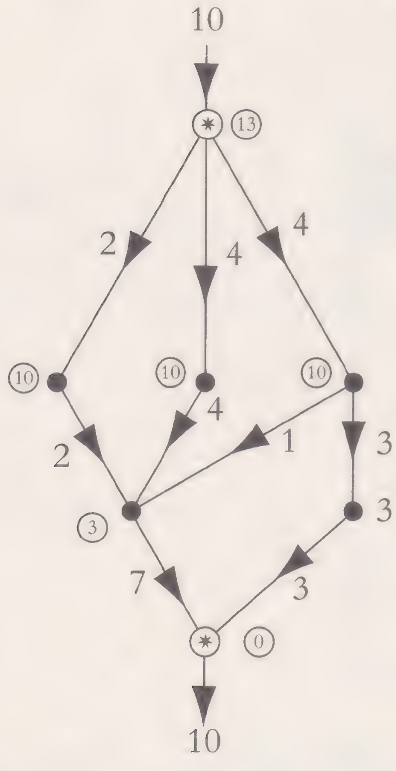


Such graphs are used in places of public interest to show the level of accessibility to various city departments to ensure and the best provision of services.

Once a graph of connections and a sketch to scale have been produced, the sketch can be called an evaluated dimensioned graph with the criteria given overleaf.

Note that our examples are very simple. Where these graphs have particular interest is when they are complex, and the simplicity of analysis is welcomed.

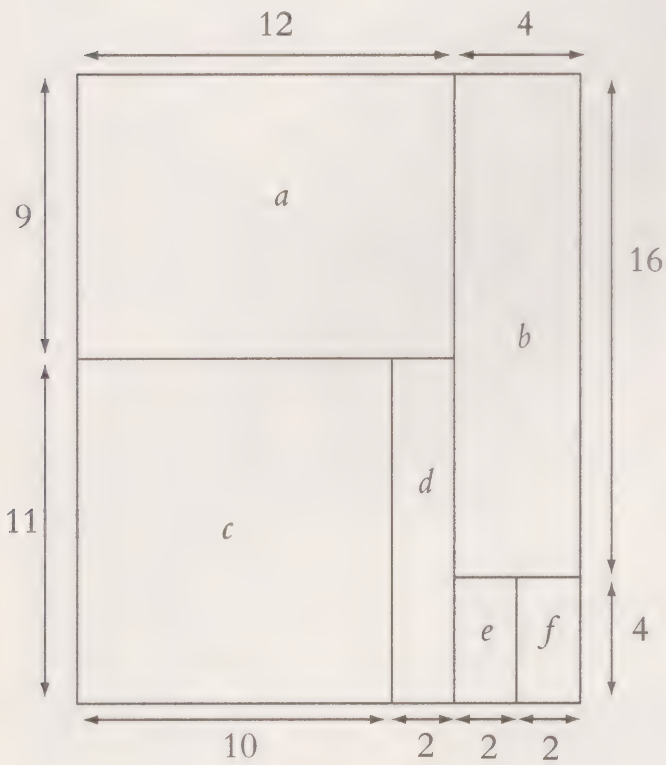


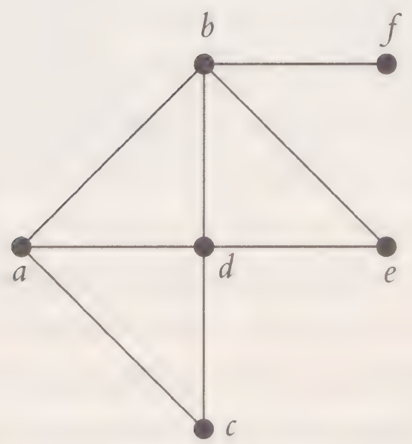


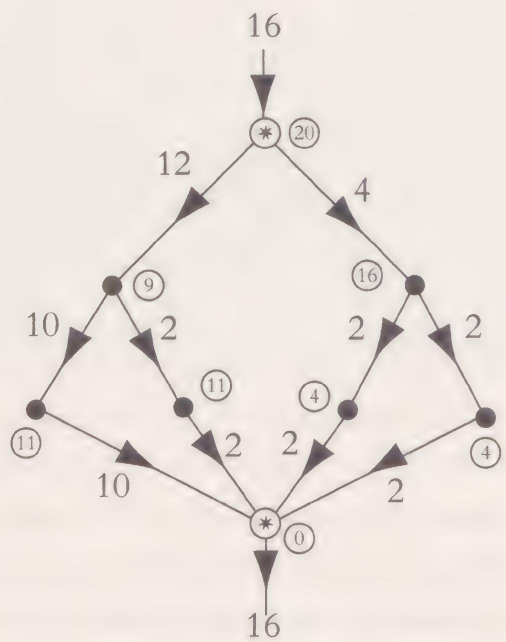
The number of vertices is equal to the number of horizontal walls, plus two special vertices (at the beginning and end), scaling the vertices from the top downwards. Edges come out of each vertex (downwards), on which the dimensions of the horizontal walls is written. The vertical dimension remaining between the wall associated with one vertex and the next one down is placed in a circle on each vertex. The total horizontal dimension is placed on the initial vertex (on the entry

edge) and the total vertical dimension in an adjoining circle. On the final vertex the vertical dimension should be zero and the edge coming out of it should have the horizontal total. Note that if the graph is not correct the sum of the values coming out of each vertex is not equal to the sum of values going in. One use of these dimensional graphs is to verify that the distribution of interior dimensions is correct.

Another example of adjacent objects and their dimensional graph is shown in the following diagrams.

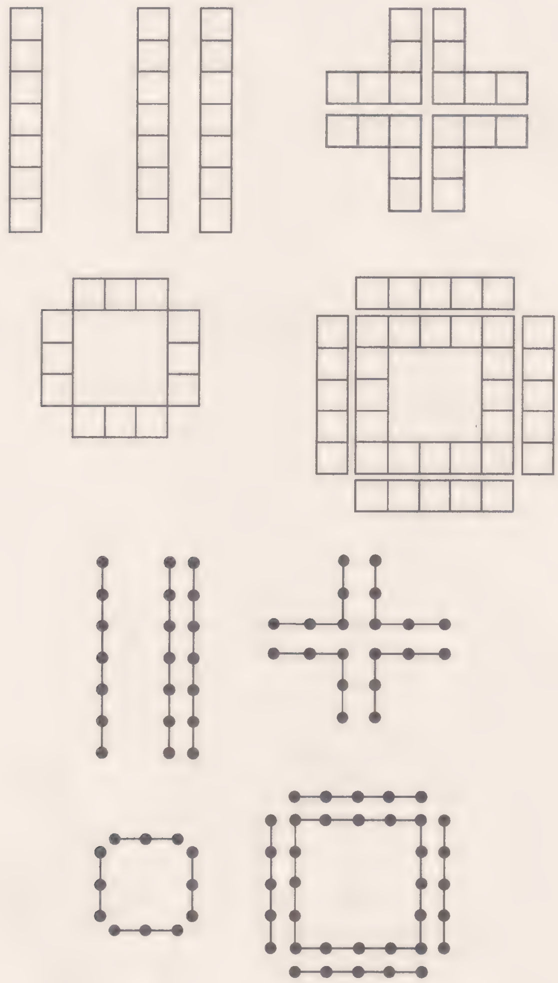






A type of evaluated graph interesting to architects is that for the theory of *graphs of effective distance* between communicated elements. That theory, specially developed by

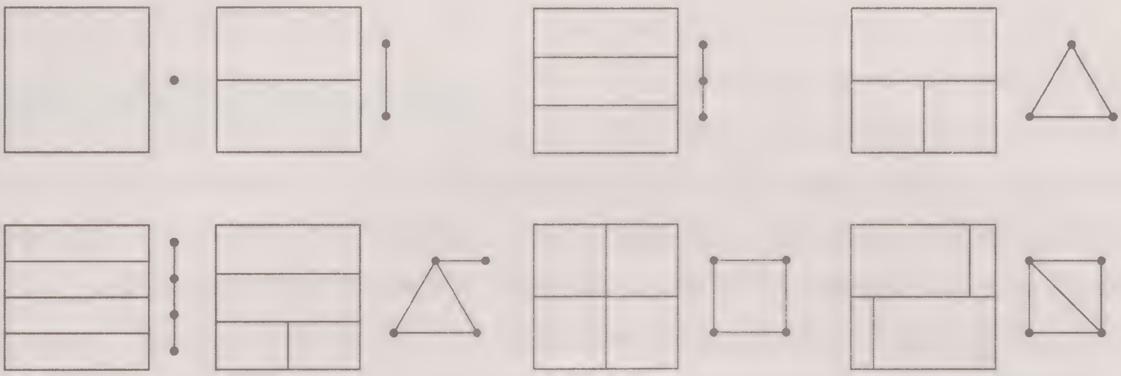
T. Tabor, can be generically described as the study of 'optimum' distributions of architectural elements that minimise route problems. Although on a small scale the problem is uninteresting, on a large scale (such as the layout of interdependent properties of a bank or similar large enterprise) the analysis of 'usual routes' can lead to a specific regrouping that allows those communications. For example, when distributing equally sized offices on a floor, the following five diagrams and their equivalent 'contiguity' graphs can be used.

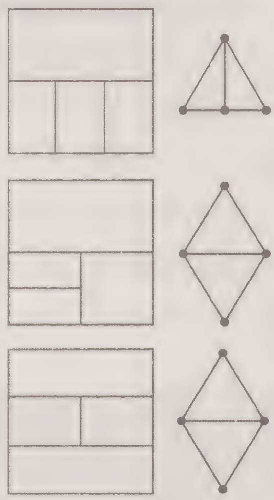


By studying all the possible distances between pairs of offices (using the real distance of the route and not the Euclidean geometric route) the criteria for minimizing displacements between on five distributions can be obtained. In Tabor's experiment assume an average speed of 1.5 m/sec. on the floor and 0.3 m/sec. on stairs. These minimum principles have been applied by city planners for large shopping centres, pedestrian islands, density in traffic networks, etc.

AN OPEN PROBLEM

An open problem on graph theory, with an architectural motivation, is how to dissect a square into rectangles (drawing only horizontal and vertical lines), determining all possible subdivisions and non-equivalents in each case.





Note that the problem is not to find all the possible finite graphs, but those that correspond to a plausible dissection plan. If n indicates the number of rectangles that we want to divide the rectangle into, for n = 1, 2, 3, 4, 5 and 6 it has been calculated that there are 1, 1, 2, 7, 22 and 117 different forms of topologically non-equivalent subdivisions. For $n \geqq 7$ the problem of finding an exhaustive description of those subdivisions is still open (for example, it is estimated that for $n = 7$ there may be around 700 solutions, for $n = 8$ about 10,000 and for $n = 9$ about 250,000, but those extrapolations are pending verification). Nowadays, this type of problem is attacked with the creation of algorithms capable of resolving the total number of possible solutions by means of computers.

Graphs in city planning

Christopher Alexander is a well-known American architect and professor who, in the 1970s, published challenging ideas about how graphs, computer problems and quantitative resources could help to rationalise the process of urban analysis and help designs new forms of architecture. His book *The Synthesis of Form* used graphs to study shapes. But his article *The City is not a Tree* was particularly important. In

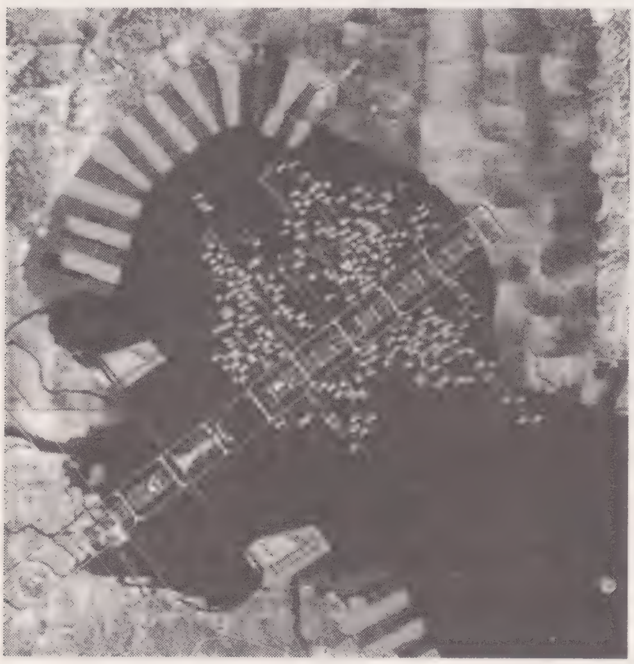
it, Alexander used the trees from graph theory as a metaphor in a discussion about urban growth, putting forward this cryptic epithet:

> "I believe that a natural city has the organisation of a semi-lattice...When we organise a city artificially, we organise it as a tree"

Alexander formulated the distinctions between natural and artificial cities based on the analysis of semi-lattices and tree structures, comparing the idea of a city to that of a complex system in which different units, sub-units and supra-units are related by a hierarchy. Alexander considers that in natural cities there is a common use of zones, objects and communications that are shared by two or more parts of the system, while in artificial cities this common use is not present, as the superposition of two units of the same type does not determine a common usable sub-unit.

An example can throw light on these distinctions. In a century-old university located in the centre of a city, the students and professors' libraries, shops and housing are usually found near the university, but mixed with other city buildings: university life is a constant interaction with normal city life – shops, traffic lights, parks, etc. are used by the whole community. In modern universities the university campus is usually organised as an autonomous area. This implies the subdivision of the campus into a residential area, a commercial area, a teaching area, etc. University life is thus submitted to a hierarchy of spaces, a distinction of usage and isolation from communities that do not share a common physical space.

Examples of typically aborescent city projects are Abercrombie and Forshaw in the planning for Greater London; and Kenzo Tange in Tokyo, Lúcio Costa in Brasilia, and Le Corbusier in Chandigarh, etc.



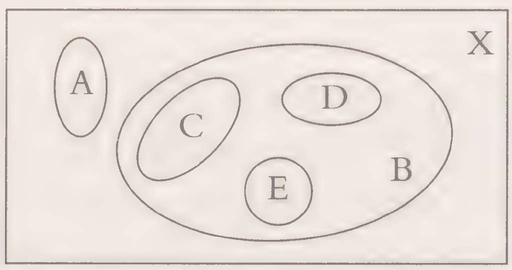
The bay of Tokyo according to Japanese architect Kenzo Tange's project (1960).

Ultimately, Alexander concludes that architects should look for city structures more complex that the tree:

> "For the human mind, the tree is the easiest vehicle for complex thoughts. But the city is not, cannot, and must not be a tree. The city is a receptacle for life."

Graphs in social networking

Graphs are also a valuable instrument in social sciences, especially in research in sociology, anthropology, geography, economics, communications, and social psychology, among others, which analyse social networks – a social structure that is represented by nodes of a graph (people, organisations, communities, groups, etc.) and the edges between them symbolise the pertinent relationships (organisation, economic dependencies, decision levels, etc.).

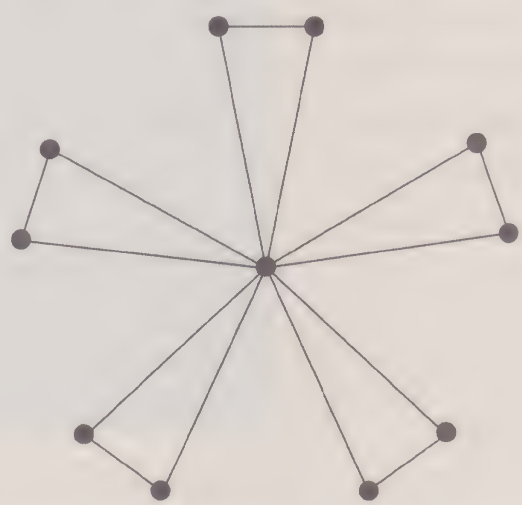


Social networks are often complex, and therefore the corresponding graph allows problems regarding relationships between activities, company groups, neighbourhoods, etc. to be visualised and understood. Nowadays Internet social media sites, and corporate intranets are also tackled in this way.

The idea of studying social networks was taken up in the 19th century (by Émile Durkheim and Ferdinand Tönnies) and was developed further during the 20th century in Georg Simmel's ideas. The pioneering studies took into account issues such as working relations between groups or people in a company, urban distribution problems, relationships between cultural communities, etc. In the second half of the century groups at the universities of Harvard (Harrison Whitie, Talcott Parsons), California (Linton Freeman), Chicago and Toronto, made many breakthroughs in this field. Among the applications of this kind of analysis of social networking we can find research on the spread of diseases (AIDS, malaria, tuberculosis), the dissemination of new ideas, the analysis of the impact of social policy, and even the spread of rumours and opinions.

THE POLITICIAN'S FRIENDS

The following problem has been doing the rounds in mathematical circles for years: let's suppose that in a group of people (at least three) any pair of people has exactly one friend in common. Then there is always one person (called 'the politician') who is friends with everyone in the group. Paul Erdös, Alfred Rényi and Vera Sós formalised and resolved this problem using graphs: if in one graph there are n vertices ($n \geq 3$) and for any pair of vertices there is a vertex adjacent to them, then there must be a vertex adjacent to all the rest.



From the graphs that help to visualise the social network being analysed, quantitative evaluations are introduced, many of them backed up by computer programs, studying parameters such as levels of dependency and proximity, levels of centrality, flows between nodes, cohesion, equivalence, etc. For example, the structural cohesion is the minimum number of members which, if they were removed from the group of the network being analysed, would leave the group disconnected from the network. The intensity of relationships, probabilities of passing on information, frequency of interaction, separation between nodes, etc. Thus, the study of connectedness is an important tool for resolving key functions in an organisation (information transit, hierarchies, leadership, etc.). The calculation of influence indices is another interesting method, be it at a political or commercial level.

Stanley Milgram's 'small world'

In 1967 psychologist Stanley Milgram completed the so-called 'small world experiment', which consisted of selecting a sample of citizens and asking them to deliver a message (a letter for example) to some specific – yet unknown – recipients with the help of people they knew, who would then pass the message along a chain until it reached its destination. It turned out that with just six steps the letters reached the intended person. This experiment has been repeated many times and it appears to be confirmed that the number of steps in the chains is always very small (five, six, eight, etc.).

Graphs and timetables

In a complex world such as ours one the of crucial issues is the need to efficiently plan all sorts of timetables in order to optimise time. "Time is golden" after all.

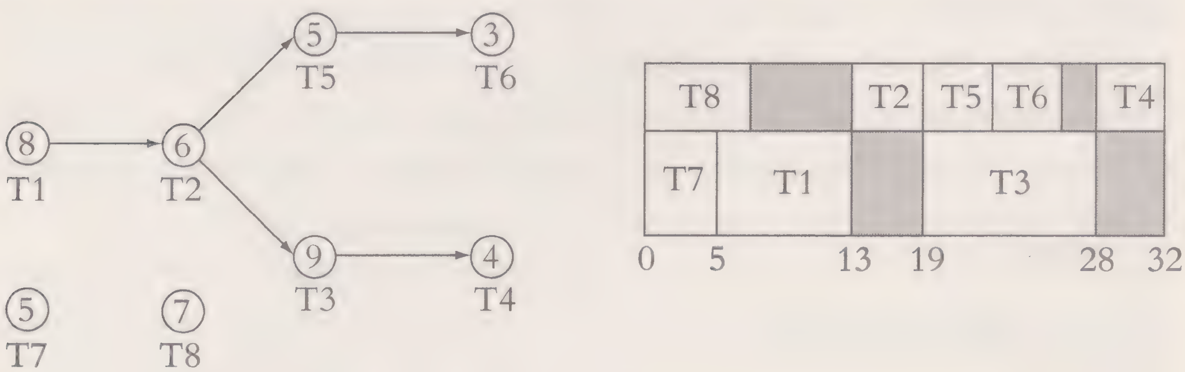
The reason behind this constant search to optimise time is to get the most out of workers, or machines involved in transport, production, providing services, etc. Reference has already been made to complicated situations such as minimising the time between the arrival and departure of a plane or planning architectural work. Here we would like to demonstrate how the subject of graphs and time also has an application in situations that are perhaps closer to our own lives.

Think of an everyday activity like buying various ingredients, preparing a meal and then serving it. How do you organise this activity? You should perhaps consider the following plan:

1. Number all of the tasks and evaluate the time required for each of them.
2. Analyse the tasks that are independent from one another (like the shopping for example) and those which need to be done sequentially, in a specific order. At this stage, a graph could help. Include the tasks with their times as vertices and directed lines (arcs or arrows) as edges indicating their order.
3. So to optimise the time based on the number of people who are going to collaborate and the number of available machines (ovens, mixers, pots, etc.), apply an algorithm that allows everything possible to be done in parallel (preparing the table, etc.) and the rest in sequence.

If you remember the greedy algorithm which you will have seen in colouring graphs, you can try to apply it to this gastronomic adventure, in which investing the minimum amount of time is the main objective: assign the tasks taking into account their numbering and sequence and the shortest times.

In this diagram you can see a generic example of tasks and times: a directed graph and a programming list assuming two machines acting in parallel.



Because some of the necessary tasks take longer, one way of programming them is to follow the decreasing time algorithm, in other words, give priority, while respecting the sequencing, to whatever takes the most time to execute (boiling the water, roasting the turkey, etc.).

As you may have realised, all this is a very important issue in fields as diverse as car, television, computer productions lines, etc.; printing and copying shops with various machines and employees; planning surgery time in a hospital, operation waiting lists and doctors' timetables; the distribution of songs between two music CDs; the organisation of timetables and holidays in companies with several shifts; timetables in hotels and restaurants; and bus, train and plane timetables.

MATHEMATICS AND FRIED EGGS

One of the most popular stories regarding the ways mathematicians think (and act) makes reference to how these professionals try to optimise everything they do in their life, following algorithms which they use even when performing the most insignificant tasks. A mathematician explained the process for cooking a fried egg in great detail: take the frying pan out of the cupboard, light the gas, put the pan on the heat, pour some oil in, add the egg, fry... The question which is then put to the mathematician is "And what would you do if the gas was already lit and the pan was on it?" The mathematician's response was: "Take the frying pan, put it in the cupboard and proceed as per above."

In all these cases the goal is to optimise time, with everything that means for costs, quality of service, efficiency of the organisation, etc.

Naturally sometimes it is not the time we are worried about but other factors such as space. Packing suitcases, loading furniture onto a removal van, preparing a container for sending goods somewhere, etc. – these are all problems in which graphs and the associated algorithms could be of service for optimising space. Thus, an *algorithm for the next decreasing possibility* would be a heurisitc way of packing the largest thing first.

NP-complete problems

All the algorithms described here for optimising time and space, as with the case of the previously mentioned travelling salesman, are difficult to implement when confronted with a certain complexity of data and agents. It cannot always be guaranteed that the proposal the algorithm leads us to is the best possible solution. As with all problems categorised as *NP*-complete, it appears impossible to find algorithms for fast solutions. We should trust our ability to resolve problems, in each case searching for the best we can do, without relying on the appearance of efficient algorithms that a machine can carry out in a reasonable time scale.

In 1900 David Hilbert, at the International Congress of Mathematics in Paris, proposed a list of problems whose resolution he considered to be the most important in the 20th century. One hundred years later, for World Mathematics Year, the Clay Mathematics Institute in Boston announced prizes of one million dollars for anyone who could resolve any of the so-called Millennium problems. It is worth

German mathematician David Hilbert.

noting that this prestigious institute was funded in 1998, by Landon T. Clay, who is a well-known businessman and admirer of mathematics, which is possibly why he offered a tempting amount of money for the pending problems. In contrast Hilbert could only offer eternal fame to those who resolved his problems.

Of the seven big Millennium problems, number one is the so-called $P = NP$ problem. This problem falls within the context of what is known as the theory of computational complexity, and involves analysing computation times necessary for resolving a problem.

It is possible that, due to the million dollars or simply because of noble desires for progress, this search for algorithms in graphs also helps to motivate the development of new forms of computation, beyond the possibilities currently provided by digital calculation. So-called quantum computing, which is currently only a theoretical field, could in the future open new effective possibilities in computation, 'exponentially' expanding the current limits. As always, the most interesting is yet to come.

Recreational graphs

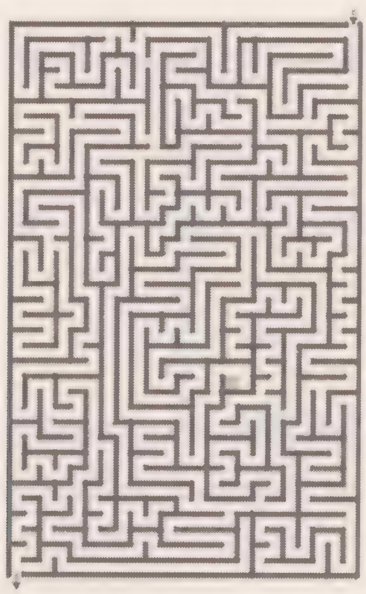
There are many games that involve drawing graphs, or those which by means of graphs analyse whether there is a winning strategy or not. As a sample and to mark the end of our journey, here are a few historic games.

Who will say 20?

The first player says 1 or 2. They then take it in turns and can add 1 or 2 to the previous result. Whoever says 20 wins. Is it a game with a winning strategy? What if the target was 83 or 100 instead of 20?

The maze in Rouse Ball's garden

Rouse Ball helped to popularise many concepts thanks to her entertaining writing on recreational mathematics. In the famous Ball maze there is the entry/exit at the top and the treasure is marked a spot. Can it be reached? Have a go before looking at the solution. Did you find it? The itinerary given has lines and junction points. Each edge must be travelled twice (going in and coming out). With vertices of an even degree this is possible and it is sufficient to mark indications of where we have already been on the floor so that we do not repeat a dead-end section.

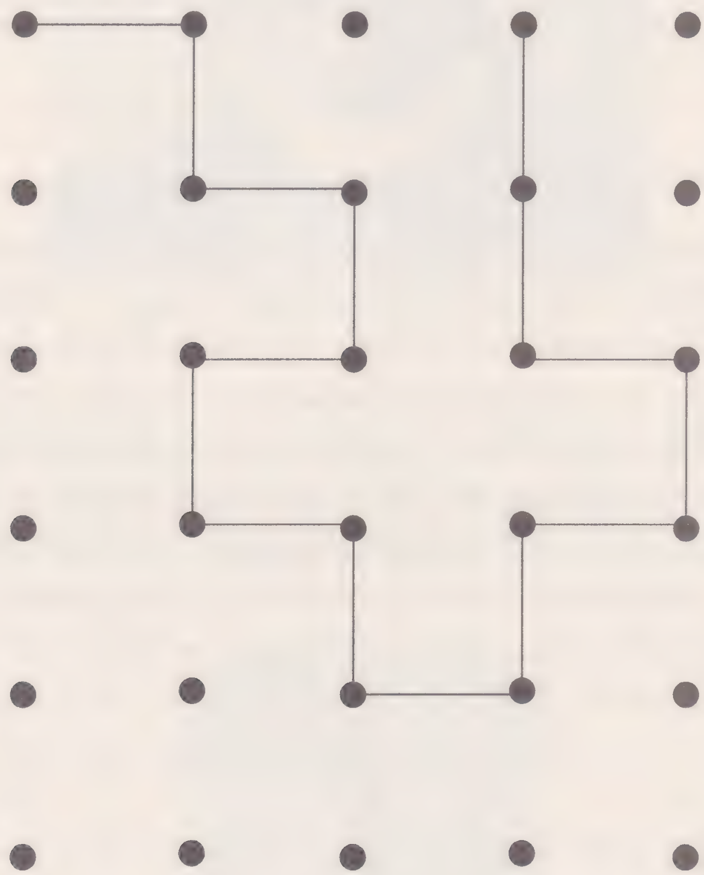


Rouse Ball's maze.

The snake game

Invented by David L. Silverman, the snake game consists of a rectangle of 5 by 6 points (or any other amount) in which the players draw a unit segment starting from any point (no diagonals allowed) forming a continuous path, being able to add segments at either end of the path. Whoever is forced to close the path is the loser.

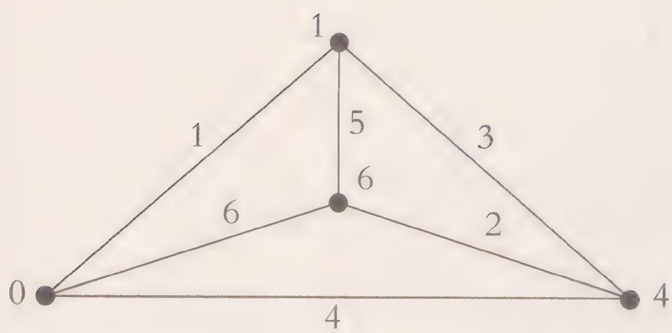
Is there a winning strategy?

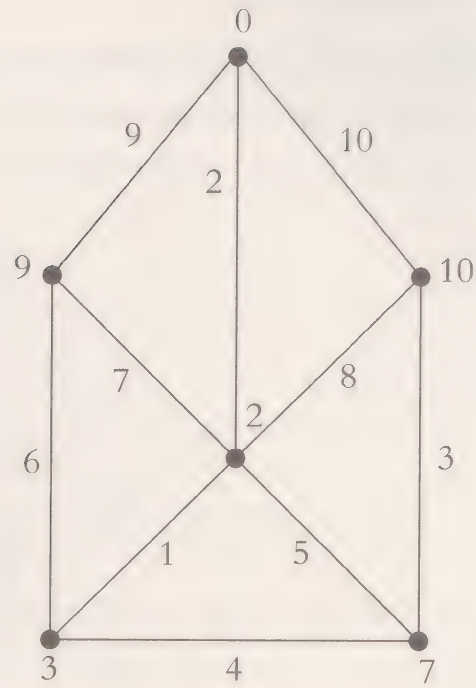


The elegant numbering of a graph

This game from Solomon W. Golomb consists of drawing a graph and assigning zero and other different positive integers to its vertices.

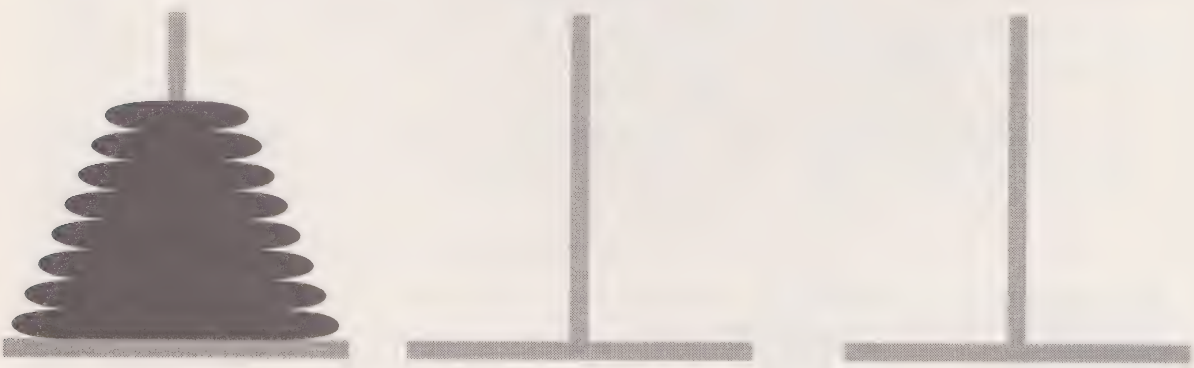
The game must be solved in such a way that the difference in values between two adjacent vertices on the edges are all different and, at the same time, making the highest number assigned to the vertices as low as possible.





Towers of Hanoi

Invented by Eduard Lucas in 1883 (and shrouded in false legends), this game consists of three vertical dowels, the first of which contains *n* different discs (with a hole in the middle) placed in ascending order from largest to smallest. A disc can never be placed on top of a smaller one. The idea is to move the discs around on the dowels and build the same tower on the third dowel. Only one disc can be moved at a time and it must be placed on top.

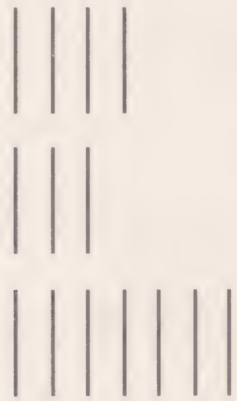


The starting position of the Towers of Hanoi

The number of solutions for *n* discs is $2^n - 1$, a number which grows very quickly. You can use graphs to help see patterns in the movement. There are also online versions of this game.

The NIM game

Two players place various separate groups of tiles in lines. The first player takes between 1 and all the tiles from a line. Then the second player takes tiles from the remaining lines. And so on, taking it in turns. Whoever picks up the last tile wins.

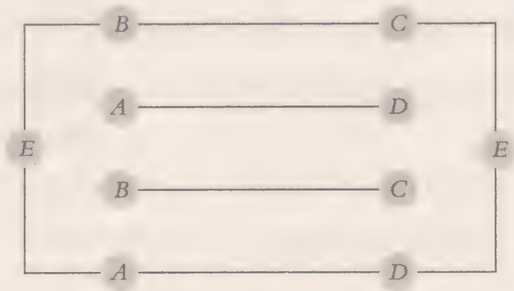


Two circuits from Martin Gardner

Fascinated by planar graphs, Martin Gardner proposed and solved numerous problems to the delight of his readers around the world. Thinking of publishing the application of planar graphs in circuits, Gardner already argued that this case of circuits was a good example in which the unions between the different points (vertices) should be made between lines which form a planar graph, avoiding cross overs that cause short circuits. The reader is invited to solve the following challenges (before checking the solutions at the end of the chapter).

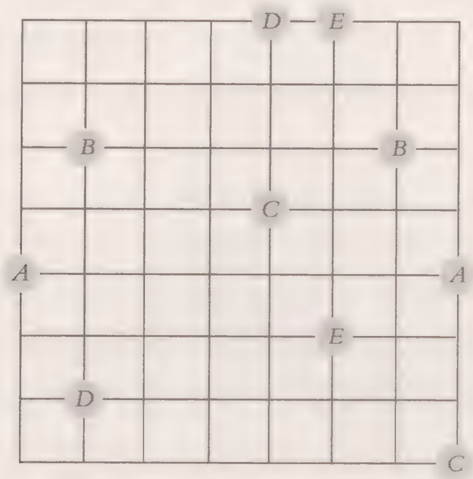
The circuit in a rectangle

In this rectangle (and without going outside it) draw five continuous lines joining *A* with *A*, *B* with *B*, *C* with *C*, *D* with *D*, and *E* with *E*, without crossing either of the segments *AD* and *BC* marked on the diagram.



The circuit on a grid

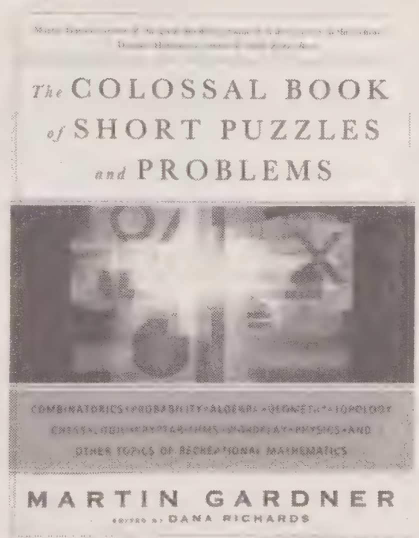
In this 7 × 7 grid, five lines must be drawn between each of the pairs of points with the same letters by only following the segments of the group of squares and never crossing each other.



You are invited to put your thinking cap on and try your patience in a search for the only solution to the problem (before racing to the back of the book for the solution).

MARTIN GARDNER (1914–2010)

In the canopy of stars of scientific dissemination, the figure of American Martin Gardner shines brightly. Born in Tulsa, Oklahoma (USA), he studied philosophy, but after graduating he worked as a journalist. For many years (from 1956 to 1986) and through his monthly columns called "Mathematical Games" in the *Scientific American* and his celebrated books, he popularised all types of mathematics, games, algorithms, paradoxes, applications, puzzles, etc. He also wrote about philosophy, scientific investigations in diverse fields and was an illustrious book critic. Strangely, Gardner never carried out public work by giving conferences or courses and focussed on the task of writing his ideas down.

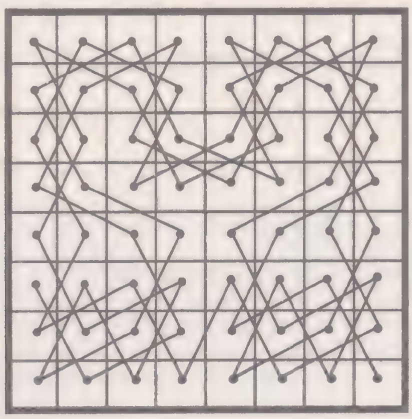


Front cover of one of Martin Gardner's numerous publications.

Knight routes in chess

The popular chess board has given rise to many mathematical challenges. A classic problem is to start with one of the pieces (the pawn, bishop, king, knight, rook, etc.) and study the routes it can make around the board, while of course following the piece's predetermined rules for movement. In the case of the knight, the question "is it possible in chess for a knight to travel around the board starting from one square and returning to it having passed through all other squares (64) just once?" is particularly interesting.

The answer is yes and the good news is that there are many possible routes. This problem, along with many other chess problems, can be studied using graph theory. Each square represents a vertex of the graph and each movement of the knight is a line that joins two vertices of the graph (respecting the knight's particular movements) and, therefore, the challenge is to find a Hamiltonian route that starts and ends on the same square.



Complete knight's route in chess.

But the restless mathematicians' imaginations were put into gear and starting with the 8×8 board they immediately considered the possibility of other boards: $5 \times 5, 6 \times 6, 3 \times 10\ldots$, and the knight problem, or that of any other piece, can be redefined on these new boards. So the subject of Hamiltonian circuits on graphs where $n \times m$ vertices has been solved. For example, for 6×6 there is a solution, but not for 5×5 or 2×8.

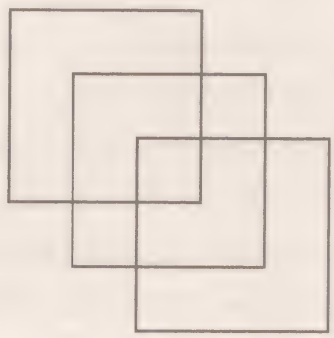
The reader could spend quite a while, even with the rook, trying to find routes from one corner of the table to the opposite diagonal, passing through all the squares on the board for the case 7×7 or in general $n \times n$.

With a simple chess set you could spend a magnificently lazy summer full of challenges and you would still need more time.

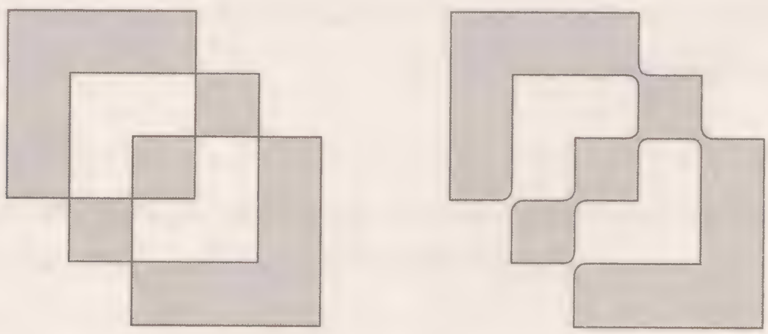
Lewis Carroll and Eularian graphs

Charles Lutwidge Dodgson (1832–1898), better known as Lewis Carroll, not only wrote *Alice in Wonderland*, but also was a great fan of recreational mathematics. He liked to propose ingenious problems for children to solve; among them he proposed some that we would now classify as graph theory problems (although in his time it was just considered a challenge of drawing a particular image without removing the pencil from the paper and passing over each line just once).

Carroll's most popular graph is that of three trick squares such as those in the diagram below. See if you can trace this drawing with a pencil before you read on.

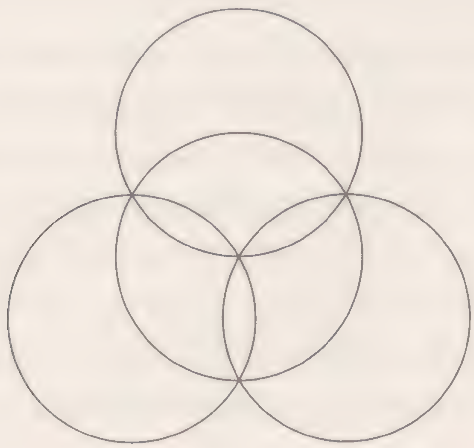


Thomas H. O'Beirne came up with a wonderful method for solving this type of problem, which consisted of colouring alternate areas (see the figure below) and then 'separating' the areas at the vertices in order to 'discover' the route. Given the outline of the route it is now easy to solve.



The four circle problem

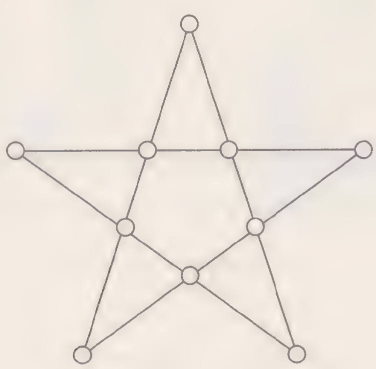
It occurred to O'Beirne years later to design a challenge similar to Carroll's, but changing the three squares for four circles that intersected each other in a wonderfully symmetrical manner as shown below.



See if you can find the route that passes through all the arcs of the four circles only once. Evidently the trick of colouring that we just mentioned for Carroll's problem could help to inspire you. If you start to get desperate after 25 attempts you can resort to the solution at the end of the chapter.

Magic stars

The so-called 'magic stars' constitute a game which never fails to surprise and where numbers and graphs are mixed, forming part of what has come to be known as recreational combinatorics.



Take a look at the pentagram above, in which the Pythagorean pentagonal star appears with ten vertices marked as circles. Is it possible to place the numbers 1 to 10 on the vertices so that all the lines of four add up to the same number?

If it is possible, the sum that is repeated in all the lines is called the 'magic constant'.

Would you like to try and distribute the numbers in the pentagram before continuing? What must the magic number be for the pentagram?

Don't worry. You can't find a solution because there isn't one. First, note that the sum of 1 to 10 is 55 and, as each number should appear in two lines of the pentagram, the total sum of all the lines would be the double of 55, or 110. Therefore, the magic number must be 110/5, or 22. Now we just need to distribute the numbers that make these sums of 22 possible in each line.

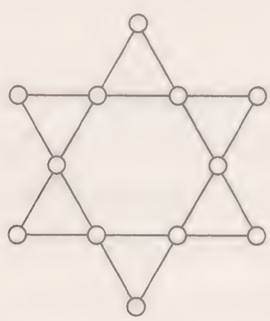
Ian Richards observed the following: each of the lines that passes through the vertex where the 1 is must contain three numbers that add up to 21 and all six must equal 42, then the 10 must be on one of these lines with the 1 (as the other six numbers without the ten would equal a maximum of $4 + 5 + 6 + 7 + 8 + 9 = 39$). If A is the line with the 1 and the 10, B is the other line with the 1 and C the other with the 10, then there are four possible combinations for A. The combination 1, 10, 4, 7 would make it impossible to form B and C. That leaves three cases:

A	B	C
1, 10, 2, 9	1, 6, 7, 8	10, 5, 4, 3
1, 10, 3, 8	1, 5, 7, 9	10, 6, 4, 2
1, 10, 5, 6	1, 4, 8, 9	10, 7, 3, 2

But B and C must have a number in common and this is not possible in any case. Therefore, it is demonstrated that the pentagram case is impossible.

The magic hexagram

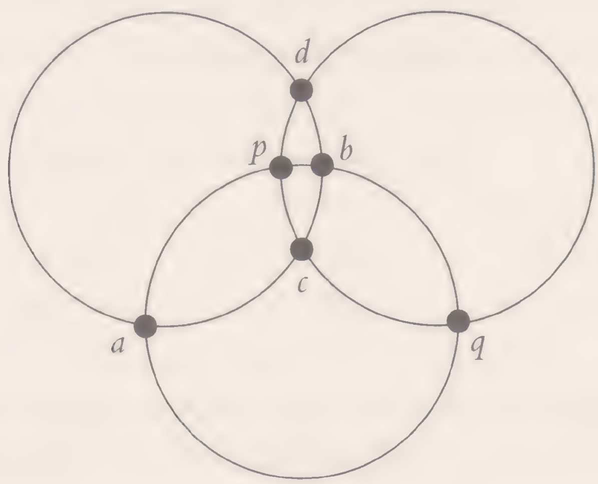
Now let's consider the magic hexagram. It is the mythical star of David or seal of Solomon, an intersection of two quadrilateral triangles.



As can be seen in the diagram there are 12 vertices distributed across 6 lines of four, so the challenge is to distribute the numbers from 1 to 12. The magic constant will be, given that the sum of 1 to 12 is 78, 78 × 2/6, or 26. Sharpen your pencil, put your thinking cap on and get ready to find a solution to the magic hexagon – there are dozens. The solutions are provided at the end of the chapter.

If, due to the excitement of success, you are starting to show signs of addiction to magic stars, you can draw a magic septagram or the octagram, find the magic constants and find some of the many possible solutions to those stars.

A simpler alternative, allowing for a more systematic form of resolution, is the case of magic circles: several circles are provided with all their possible intersections and the object is to distribute numbers so that the vertices on each circle add up to the determined quantity, for example, 20. The following diagram shows three circles with the letters *a, b, c, d, p, q* and from them we can write the relationships that must exist between these letters.



This gives us a system of equations:

$$a + b + c + d = 20,$$
$$c + d + p + q = 20,$$
$$a + b + p + q = 20,$$

and adding the three equations together gives:

$$2a + 2b + 2c + 2d + 2p + 2q = 60,$$

GAME THEORY

Graph theory is very often used for analysis in game theory. Game theory was founded by John von Neumann and Oskar Morgenstern to provide new models for economic problems. Its mathematical impact has been much wider with uses in social science and politics, marketing, finance, psychology, etc.

Upon its initial inception, the creators of game theory themselves correctly thought that "the typical problems of economic behaviour become strictly identical with the mathematical notions of suitable games of strategy". From this metaphor it was possible to develop an analysis of games with one or several players, introduce utility functions, analysis of various types of strategies (conservative, winning, risky...), computer evaluations and their uses, the issue of coalitions and voting, probabilistic analysis, processes involving luck, etc.

As, in general, the number of "players" (investors, businessmen, banks...) is finite and the number of moves, strategies or alternatives is also.

or the equivalent:

$$a + b + c + d + p + q = 30.$$

Deleting each of the three first equations from this one gives:

$$a + b = c + d = p + q = 10,$$

$$a = 1, b = 9, c = 2, d = 8, p = 3, q = 7$$

Graphs and education

Throughout the 20th century the great development of graph theory and the huge quantity of its applications to the most diverse problems has assured an educational interest in the theory that is greater than the stipulated levels.

Courses on "Graph theory and its applications" now form part of the study of mathematics, operative research, discrete mathematics, several engineering speciali-

ties (organisation of construction works, building, electrics, telecommunication, etc.) and, of course, they are present in all computer studies.

What is still pending is the educational use of graphs at pre-university levels. It is not a case of providing a chapter on graphs or elevating the theory to the same level as arithmetic or geometry, but different experiences in mathematical education show that there are resources in graph theory that do help with learning and, therefore, it would be worthwhile incorporating them.

Among the educational virtues of graphs are:

1. Graphs are often wonderful examples of *mathematical modelling*. Despite their simplicity they provide real and interesting situations that can be described and studied by associating graphs.
2. The graphs offer beautiful examples of mathematics in everyday life and, therefore, they contribute to visualising the mathematical world's presence in everyday reality, allowing *connections* to be established, which is very important.
3. Working with graphs promotes learning by *forms of reasoning* which are genuinely mathematical and have high educational value. Examples include inductive, combinatorial, spatial reasoning, etc.
4. Graphs, be they recreational or applied, allow *problems to be solved*. Thanks to George Pólya's contributions we know that resolving problems should be one of the driving forces behind the teaching of mathematics.

Having said all of this we could re-read a well-known passage from *Alice in Wonderland* in which Lewis Carroll creates a surprising dialogue between Alice and a cat:

"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.
"I don't much care where," said Alice.
"Then it doesn't matter which way you go," said the Cat.

The path to education should provide quality training for everyone and ensure that everything that is taught is up to date and applicable. It is not possible that all

official curricula remain limited to subjects from the last century or from three centuries ago – they must include themes which, if they are educational, are truly up to date.

Graphs and neural networks

The development of computer sciences has meant that there are many mathematical models aimed at achieving automatic processes (done by machines) that help human process. But given the tremendous complexity of thinking, artificial intelligence models have also had to provide non-trivial ways of proceeding. Calculations are very easy for a machine, but recommending an alternative from among several is something much more difficult.

In an early study in the development of artificial intelligence, the so-called 'expert systems' received a lot of attention. They are programmes which, by incorporating large amounts of information obtained from human experts, allowed certain solutions or decisions to be reached. Expert systems in medicine are used to aid diagnosis from a few determined parameters and the experiences of many doctors and cases, and have been particularly successful.

Other alternatives have arisen such as the genetic algorithms (inspired by developments in biology), in which random systems processed statistically already affected the algorithms and steps arranged for solving a specific problem (evolutionary programming, genetic programming, etc.). In those algorithms graphs provide a suitable language for visualising processes. In turn, these algorithms, applicable to all types of designs, systems, networks, predictions, etc., have also shown interesting relationships with graph theory studies, game theory, logic, etc.

In artificial intelligence, ideas concerning neurons and their functioning served as a metaphor for creating a new theory which is today known as the theory of artificial neural networks or simply neural networks.

A neural network consists of units called neurons, which receive a series of inputs and then emit outputs or results. There are various interconnections and the neurons can be grouped by layers. Propagation functions or calculus can be used in specific processing of the input values. (These propagation functions can be modified and transferred to specific sets of values.) In normal programming a specific algorithm calculates the possible results in an orderly way from the inputs. In neural networks the objective is that the network, checking a lot of data (in the memory) can 'learn'

automatically what comes next and, therefore, adapt the results by checking what has been 'learnt'. It should be noted that together with the 'neural metaphor' the language of human learning also plays a role ('learn', 'train', 'flexibility', 'tolerance', 'self-organisation', etc.).

Think of medical imaging, recognition of hand-written texts, voice or audio recognition, the operation of power plants, business investments, mining for computer data in large databases, industrial control issues in the operation of the plant, and so on. There are numerous applications in which the theory of neural networks is of interest. Evidently, this model can be combined with expert systems, genetic algorithms and many other contributions, such as fuzzy logic.

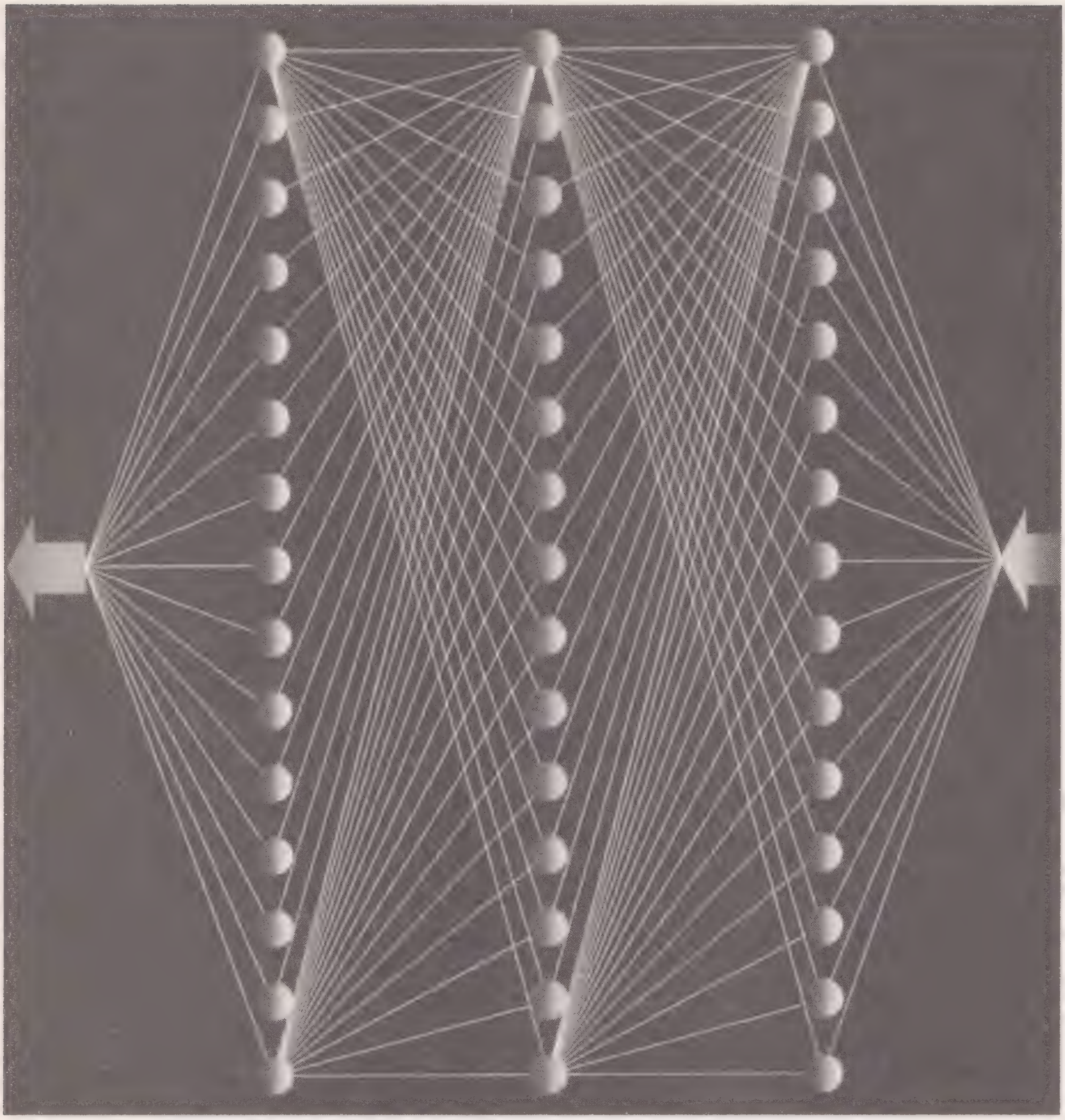
Obviously many neural networks can learn to number tables and can be visualised through directed graphs: the edges of the graph would indicate dependencies, initial and final points, interconnections, possible outputs. As with metro maps with stations and lines, these graphs help to make good descriptive maps of neural networks. An effective alternative to drawing the graph can sometimes be the collection of information on a spreadsheet.

The more neurons, inputs and interconnections there are, the greater the complexity of the process.

In the classification of neural networks, one possibility, similar to a graph, is to distinguish between forward propagation networks (those without cycles or connections between neurons on the same layer) and recurrent ones which have at least one cycle. Neural networks can also be classified by the type of 'learning' which they are capable of, or other criteria can be introduced such as the type of information (images, voices, data, etc.) that they are capable of processing.

It is very interesting that even neural networks have proved useful in mathematics itself when there is no specific model available for calculating or resolving problems, or the algorithms that exist are too complex to apply. A beautiful example of neuronal networks in mathematics, is the case of graph theory, in which these neural network techniques allow cases such as the 'traveller problem' to be tackled, where otherwise it would not be possible in a reasonable time scale.

The great advances in computer science, in which more and more sophisticated machines provide highly advanced mathematical theories, could lead us to believe that there is not long to go before most of our human skills are replaced by machines. In repetitive actions, where clear algorithms need to be applied, it is true that machines can execute certain tasks more quickly and efficiently (nu-



A diagram of the operation of a neural network, such as those which are used in computer applications, in which the inputs (the arrow from the right) are received by the receptors (circles on the right) which transmit them to the neurons (central circles). In turn, these give a response (circles on the left) which cause the relevant output (arrow on the left).

merical calculations, industrial robotics, autopilots for landing planes). But despite that, artificial machinery will never be able to replace the enviable complexity of human intelligence, which is capable of integrating matrices and overlapping information on a scale that is not programmable. In the field of robotics neural networks can help to carry out certain actions, but apparently simple tasks such as how to make the bed properly are difficult to program.

Concepts and results of graph theory are powerful instruments for approaching the organisation of complex systems. Just think about the social graphs on Facebook and Twitter, with as many vertices as friends of followers and large numbers of edges showing relationships.

Nodes, edges, grades, weightings, connections, cycles, paths, distances, components, sub-graphs, centrality, attractors, etc. – so many words and concepts from graph theory are used nowadays to solve thousands of real problems regarding networks. From railways to deliveries, from recognising patterns to creating groups of friends, from avoiding erroneous itineraries for robots to streamlining industrial production processes.

Some of the references to machines can still seem like science fiction even today. But the best... is yet to come. We'd better get ready.

Graphs and linear programming

In the 1940s 'linear programming' appeared with a bang. It was a theory that has been key in the consolidation of so-called administration sciences and which forms part of the so-called 'operative investigation'.

In planning issues (timetables, deliveries, implementation of projects, etc.), and particularly in production issues with long-reaching economic implications in complex companies, linear programming provided mathematical models, which helped to achieve the objectives (optimise benefits, minimise costs, etc).

Imagine an airline designing its route, a military brigade organising its logistics, a multinational company that manufactures soft drinks (in various sizes), NASA programming its latest adventures in space, a large telephone company laying out its lines, a telecommunications company placing routers, etc. All these bodies handle a large amount of information and are looking for clear objectives.

Linear programming has be linked to statistics, decision theory and game theory. Although at the time of its conception linear programming did not have the powerful instruments of calculation, with time the computational possibilities have given this theory great power. In fact, it is calculated that in many companies between 50% and 90% of their computational use is now dedicated to resolving computation problems.

Among the figures who have made significant contributions to linear programming are John von Neumann, Leonid Kantorovich, T.C. Koopmans, L.G. Khadrian, George Dantzig and especially Narendra Karmarkar, a brilliant researcher at the large

American telephone company AT&T Bell, whose linear programming algorithm was groundbreaking in this theory.

Mathmatician John von Neumann, a pioneer of linear programming, chats with his students at Princeton University in this photograph taken in 1947.

To capture the essence of this theory let's have a look at a short ingenious example which explains the problems we are talking about very well. Consider a company that manufactures two types of drinks, A and B, in which there are two combinations of possible fruits, a and b. The profit for each unit of A is £6 and of B is £5. For a given period, 1,000 litres of a and 3,000 litres of b are in stock. Product A mixes 0.5 litres of

GEORGE DANTZIG (1914–2005)

Considered the father of linear programming, this illustrious mathematician, professor at Stanford University for many years, carried out extensive research in the field coming up with the *simplex method*, which was essential for practical analysis. There is a popular story about Dantzig according to which, having arrived late to a Jerry Neyman class on probability, he saw two statements on the blackboard. He thought it was 'homework', so after class he solved them. Neyman was speechless. The statements had been unresolved, open problems which he had quoted. If Dantzing had known he may never have taken them on.

a with 0.5 litres of *b*, while *B* mixes 0.3 litres of *a* with 0.7 litres of *b*. Can you optimise the profit? The following table summarises the situation.

	1,000 litres of *a*	3,000 litres of *b*	Partial profit
Product *A*	0.5 litres	0.5 litres	£6
Product *B*	0.3 litres	0.7 litres	£5

In general the plan is always of the following type:

1. What are the available resources?
2. What is the available amount of each resource?
3. What are the products that have to be manufactured?
4. How is each product produced from the resources?
5. What are the unknown quantities?
6. What is the formula for the profits?

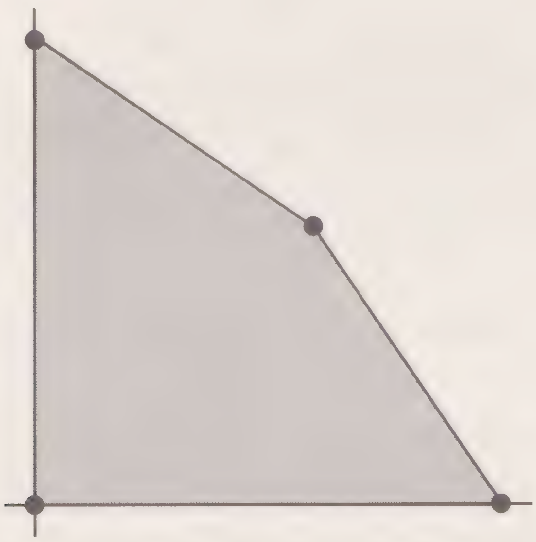
In the example, x is the number of units of A and y the number of units of B, which need to be produced with the resources a and b. The *formula for the profits* which need to be maximised is:

$$6x + 5y,$$

but variables x, y are subject to the restrictions of the resources:

$$x \geq 0$$
$$y \geq 0$$
$$0.5x + 0.3y \leq 1{,}000$$
$$0.5x + 0.7y \leq 3{,}000$$

Now that the model is complete, the problem is to find the maximum for $6x + 5y$ between the values (x, y) which satisfy the four above restrictions. Now a representation is made of the *viable region* representing all the points (x,y) of the Cartesian plane which correspond to the restrictions.



Graphical representation of the viable region, which takes on a polygonal form.

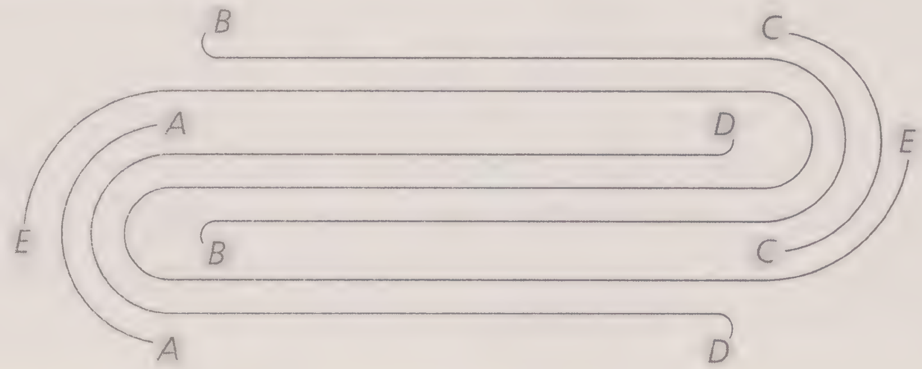
The viable region will be polygonal and it is at the corners (vertices) of this polygon where the values (x, y) which allow the maximisation of the profit $6x + 5y$ can be found. Thus, we proceed to:

1. Calculate the corners of the viable region.
2. Evaluate the profit in each of the corners of the viable region.
3. Choose the corner that produces the most profit as the production policy.

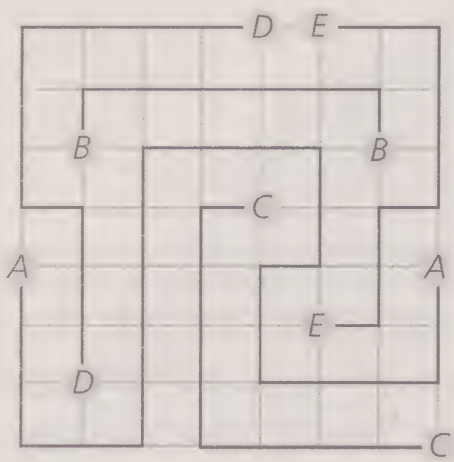
You can see that if there were a lot of products and many resources, viable regions with many corners would appear (more computation!) and that the planar diagrams are replaced by other three-dimensional or more complex diagrams. This is where

SOLUTIONS TO THE PROBLEMS

The circuit in a rectangle:



The circuit on a grid:

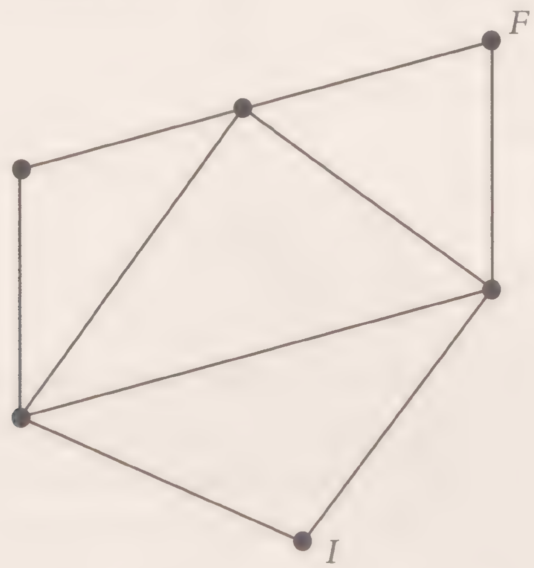


The four circle problem:

The magic hexagram: A solution to the magic hexagram is, placing the numbers in rows from top to bottom: 10; 4, 7, 9, 6; 8, 5; 1, 11, 12, 2; 3.

graph theory appears again, in the simplex method invented by pioneer George Dantzig which allowed for computer programming.

Think of the viable region as a graph (it could be a polygon on a plane, a polyhedron in space or a general planar graph).



Planar graph of the polyhedral viable region.

Instead of calculating the profit formula f in every corner, the idea is to select one of them at random and then calculate f in the adjacent corners. Once the most profitable corner has been located we again check all its adjacent corners, and so on.

The search for quick algorithms has always been a prime objective in the business world. For example, Karmarkar's contributions have allowed optimum solutions to be found with processes between 50% and 100% quicker than the pioneering simplex method.

Epilogue

The first proof of the appearance of abstract knowledge could be an engraving or a cave painting from 35,000 years ago.
Jorge Wagensberg

There are books that go straight onto the shelf and are not read. Other books are read but not kept. And there are books that are read, kept and lead to the search for more books on the same subject. We would dearly love this little guide to the world of graphs to be in the third category. In fact, there are a great many writings on graph theory and its most diverse applications (some of which appear in the bibliography) or on related fields of knowledge (topology, algorithms, discrete mathematics, etc.). We encourage you, if you are interested in the topic, to expand your knowledge.

Now that you have finished reading this small volume, beyond all the details and ideas that you have discovered, we would like to finish off by reminding you what graph theory demonstrates. With extraordinarily simple diagrams of points and lines, it is possible, by means of ingenious reasoning, to describe and resolve many problems that arise in interesting and highly varied situations. This is the revolution of diagrams, the potential of simplicity.

Reality is complex, there are many characteristics and factors that impact on natural phenomena, but sometimes the art of simplification, of ignoring details which are unrelated to the essence of the problem and concentrating only on what is substantial, is the best route to understanding the issue being analysed.

The potential graphs could have a clear parallel with the art of the 20th century. Instead of continuing to seek hyper-realistic goals or ever-growing embellishments, important painting and sculpture movements have rediscovered the artistic value of points of colours, the painting of lines, the purest geometric shapes, proving how from the essential shapes and basic colours it was possible to create new codes of expression, a new aesthetic for communicating emotions.

Graph theory invites you to maintain this vision, to focus just on the essential in a world that is so complicated.

We are going to finish this epilogue with a philosophical reference to the famous question, "Why does the real space of the surroundings in which we live have three dimensions?" Years ago now G.J. Whitrow in his work *The Structure and Evolution of the Universe* argued that in physical dimensions greater than three the perfect stability and movement of the planets around the sun would not be possible. But in two dimensions intelligent life would not be possible just as graph theory shows: the mind needs an enormous quantity of neurones (vertices) interconnected by nerves (edges) that must not cross over one another. As can be seen in planar graphs in a world of two dimensions this neural connectivity would be impossible. The following analogy from Whitrow is particularly interesting: even our own mind is imaginable as an immense neural graph.

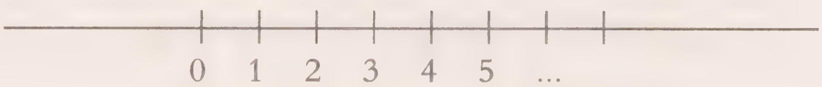
May good graphs be with you and may you enjoy them.

Appendix

Graphs, Sets and Relations

The great edifice of mathematics has, as all solid edifices do, significant fundamentals. Logic of course plays an essential role in setting deductive rules, the concepts of truth and falseness, the distinctions between axioms/postulates and theories, the admissible forms of demonstration, etc. Set theory is another fundamental pillar of the edifice, with which more genuine concepts of mathematical structures: elements, sets, relations, functions, etc. can be formalised.

In the intuitive approximation of set theory, both symbolic and graphical descriptions are used. If $\mathbb{N}=\{0,1,2,3,\ldots\}$ represents the group of natural numbers, in the following diagram the group is represented by means of points marked on a straight line.

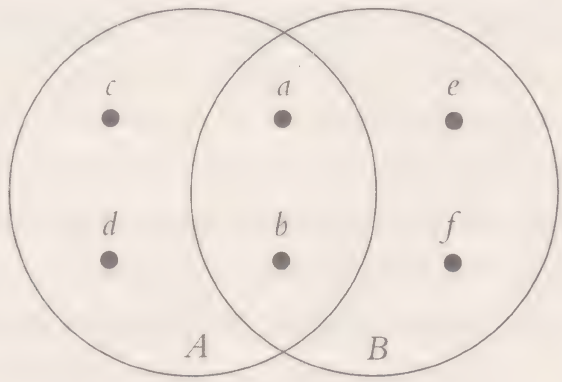


Natural numbers on a straight line

GEORG CANTOR (1845–1918) AND SET THEORY

This brilliant German mathematician created set theory in order to provide greater rigidity for many mathematical concepts and, in particular, to be able to approach the concept of infinity with some clarity. Frege and Dedekind also made significant contributions. Thanks to Cantor it could be considered that "a finite set is one that is not infinite" and a set *A* is said to be infinite if there can be a binary (one to one) relation with one of its subsets. Cantor clarified the issue of 'numberable' infinite sets (such as natural, integer and fractional numbers), establishing various categories of infinites (transfinite, cardinal and ordinal numbers). All these ideas caused ferocious confrontations with other mathematicians of the time (Leopold Kronecker was his main enemy) and gave rise to many paradoxes which had to be separated out. But the beautiful, potent and fundamental set theory was born!

For finite sets $A=\{a,b,c,d\}$, $B=\{a,b,e,f\}$ Venn diagrams are normally used where the elements are represented by scattered points and their groups are limited by closed curves.

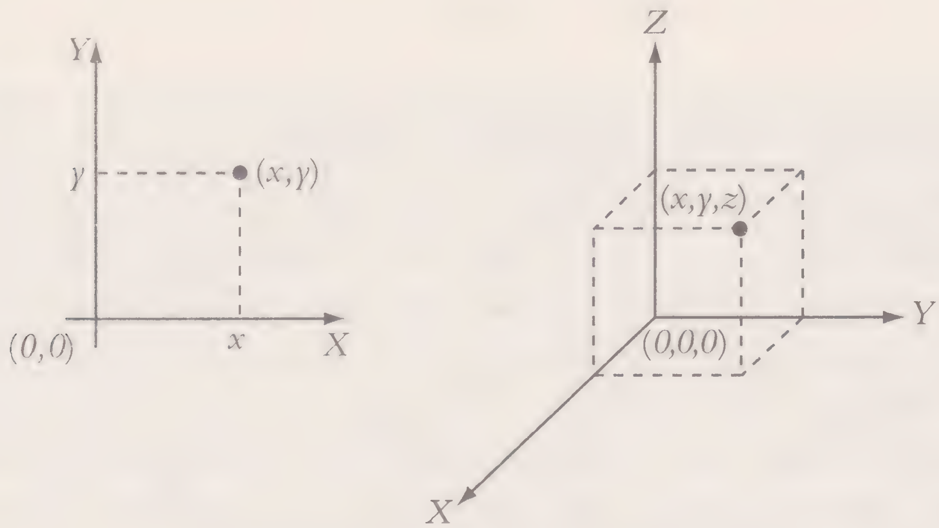


A Venn diagram.

From two sets A, B the *Cartesian product* $A \times B$ is defined as follows:

$$A \times B = \{ (a,b);\ a \text{ from } A,\ b \text{ from } B \},$$

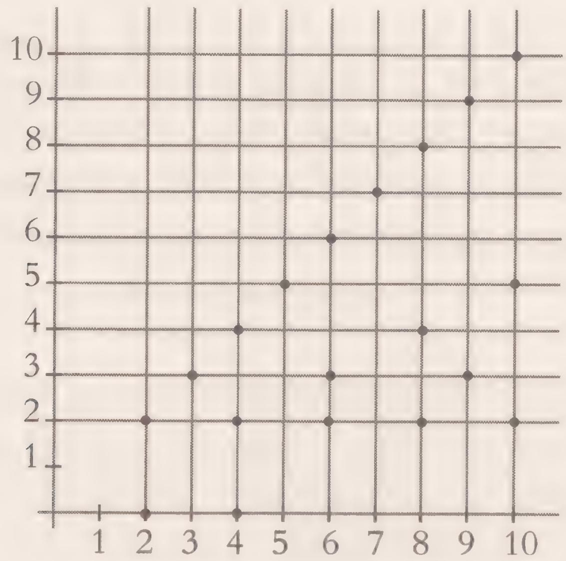
in other words, the set of all even ordinate numbers (a,b). This product is related to the tradition started by René Descartes of placing points on a plane (x,y) or in a space (x,y,z) by means of these ordinate numbers, which are the coordinates (or projections on the axes). Words are also ordinate groups of letters...



Cartesian coordinates on a plane and in space.

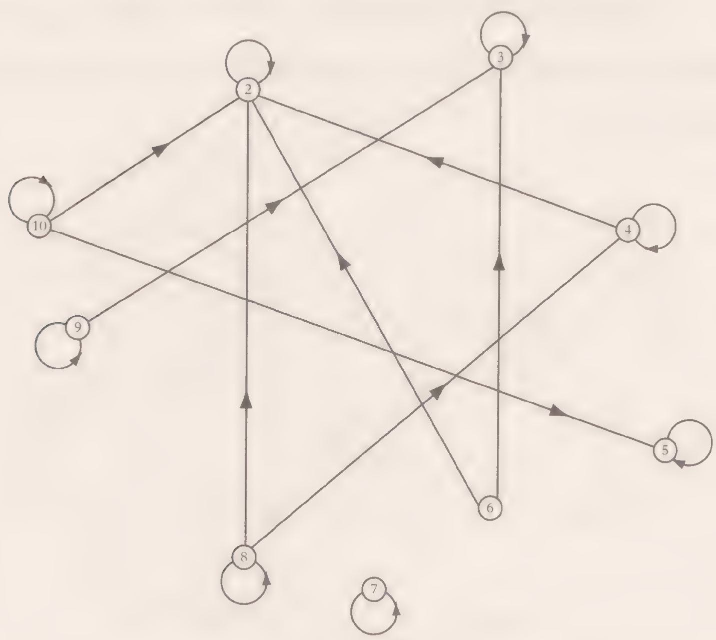
In terms of Cartesian products $A \times A$ for a set itself it is possible to formalise the key concept of *relation* R as a subset of $A \times A$; in other words, the relation indicates the elements of A which are interrelated. If (a,b) is in R, both elements a and b are

related, and if (a,c) is not in R, then a which was related to b is not related to c. Thus, given the ratio R, for each element a it makes sense to consider the *class* of all the elements related to a. If (a, b) belongs to R it is also written '$a\ R\ b$' to indicate 'their' relation. For example, consider the set $A = \{2,3,4,5,6,7,8,9,10\}$ and the relation R in A: $a\ R\ b$ if 'a is a multiple of b'. One option would be to use a Cartesian representation indicating the related pairs.



A Cartesian representation of a relation.

But an alternative would be to use a directed graph as shown here:



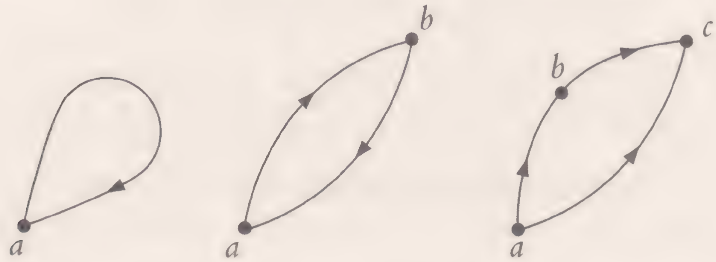
A directed graph representing a relation.

Equivalence relations

With a view to being able to make classifications in a set, so-called 'equivalence relations' R in a set A are of particular interest. Three properties are needed for them:

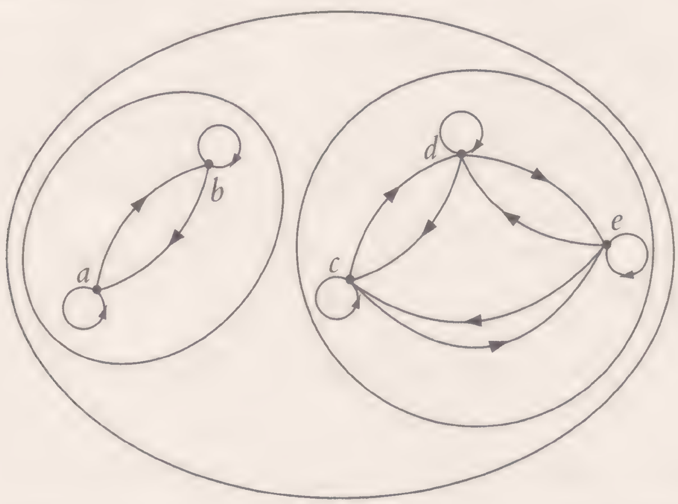
a. Reflexive property: $a\ R\ a$.
b. Symmetric property: if $a\ R\ b$ then $b\ R\ a$.
c. Transitive property: if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$.

Put into words, all elements are related to one another, there is symmetry in the relation and transitivity in the related triples. When R satisfies all these properties then set A is classified (divided) into *classes*. These relations, in finite sets, can be represented using graphs: the elements are represented by points and joined to those they are related to with directed lines.



Graphical representation of an equivalence relation.

As the equivalence relation leads to classification, diagrams such as those in the figure can be produced.



Classification associated to an equivalence relation.

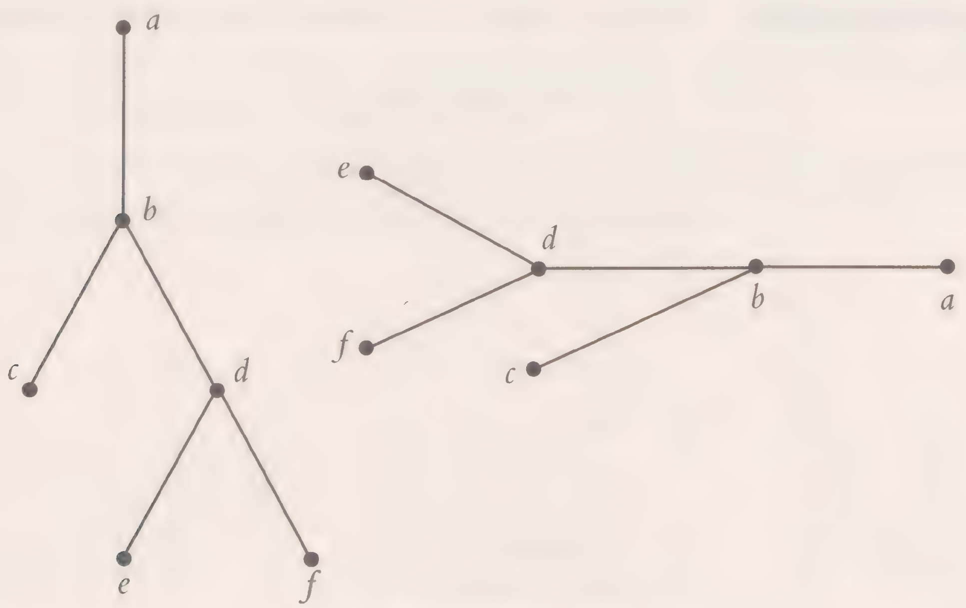
If A is a set of people and R is the relation 'having the same age' the classification allows you to consider the classes of ages. If A is the set of integers and R is the relation between numbers $a\ R\ b$ if $a - b$ is an integer multiple of 2 it will be classified into evens and odds.

Ordered relations

Another type of relation which is essential to mathematics (and to life) is *ordered* relations, which demand the following properties:

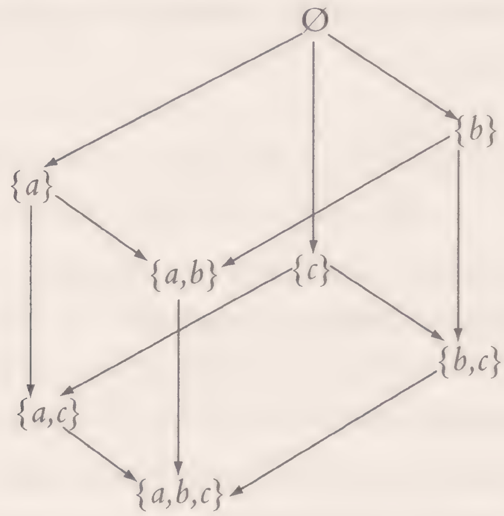
a. Reflexive property: $a\ R\ a$.
b. Antisymmetric property: if $a\ R\ b$ and $b\ R\ a$ then a must equal b.
c. Transitive property: if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$.

Instead of '$a\ R\ b$' the notation '$a \leq b$' is normally introduced, which is well-known on a numeric level $(0 \leq 1 \leq 2 \leq \ldots)$. So for each element a it makes sense to consider the set $\{b/a \leq b\}$ of all those which are greater than a or $\{b/b \geq a\}$ of all those which are less than a. Again, in terms of graphs, it is possible to introduce representations by assigning vertices to the elements, lines to the union between the ordered elements and adopting criteria for verticality ('those that are lower are less'), horizontality ('those that are to one side are greater') or use directed graphs to clearly indicate the order.



Visualisation of ordinations.

In the diagram below, with arrows to denote 'inclusion in', we can see the ordination of the parts of a set formed by three elements $\{a,b,c\}$.



An inclusion graph.

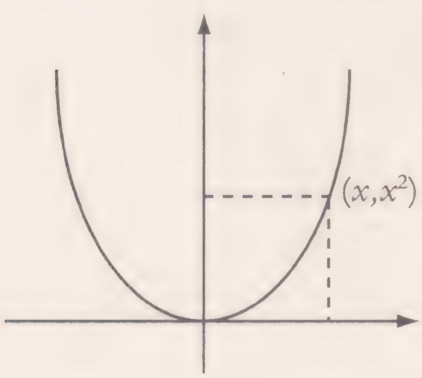
Family trees are examples of ordinations between people. The arrows can also help to highlight the order, but their presence can be substituted by the verticality and horizontality criteria.

Functions

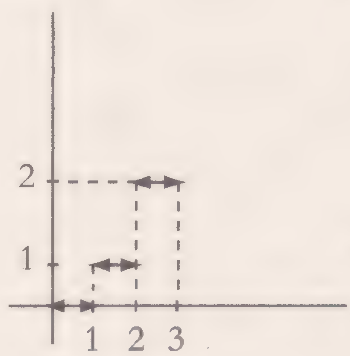
Another basic notion in set theory is the consideration of functions $f: A \to B$, where elements a from A are assigned a unique element $b = f(a)$ from B. In this case the graph of f is considered

$$\text{Graph } (f) = \{(a, f(a)) / \text{ a from } A\}$$

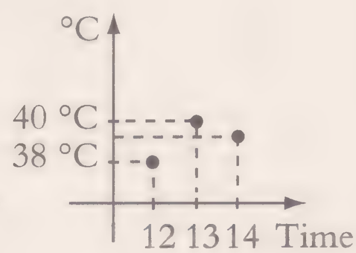
and represents that set in $A \times B$.



Graph (parabola) of the function $f(x) = x^2$.



Graph of the function of the integer part of positive real numbers.



Body temperatures

GEORGES PEREC AND HIS "THINK/CLASSIFY"

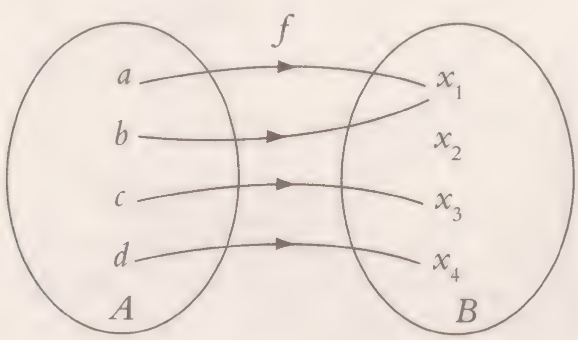
Between 1976 and 1982, provocative intellectual Georges Perec published numerous surrealist articles which had high critical content. Two brilliant articles were specifically about "Think/Classify" and "Brief notes on the art and manner of arranging one's books". In them, Perec shows how difficult it can be in our lives to classify things and people, to organise books, etc. For example, Perec shows the enormous difficulty of forming an 'organised' library, as the books could be classified/ordered alphabetically by authors' surnames, by colours of covers, by types of binding, date of purchase, date of publication, format, genres, languages... Situations that are difficult to solve may appear when going from theory to practice.

Obviously current graphical calculators and computer programs allow perfect functional representations to be drawn. But in many cases these representations are intuitive graphs or approximations.

In the first two examples, two functions, well defined by formulae, can be seen, but in the third example the information is reduced to a graph of points representing a small amount of data on temperature. How can the reasonable temperatures be extrapolated between the hours for which data has been collected? Obviously the points could be joined by straight lines, although there are other options.

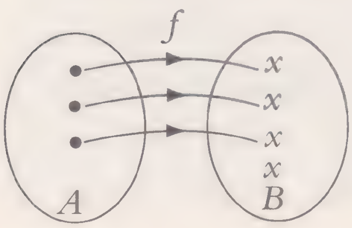
In the world of experimental data graphs with a finite number of points are very common $(x_1,y_1),\ldots,(x_n,y_n)$. The study of graphs which pass through these points, or roughly describe the distribution of them, is of great statistical interest, especially when trying to see if there is a relation between the values of one variable $x_1, \ldots, x_n$ and those of another $y_1, \ldots, y_n$.

For functions between two finite sets A and B it is also common to see a representation which combines graphs with Venn diagrams.

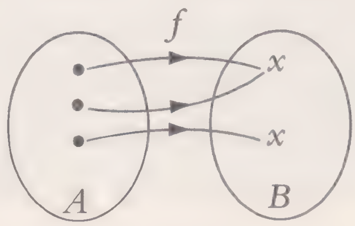


Visualisation of function f of {a,b,c,d} in {1,2,3,4}.

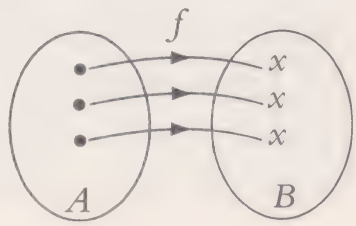
When different elements have different images, the function is said to be *injective*; when all elements of the input set are a copy of any element, the function is said to be *exhaustive* (or superjective); and when the function is both injective and exhaustive, in other words, there is a one to one correspondence between the elements, then the function is said to be *bijective*. The following graphs illustrate these three categories.



An injective function.

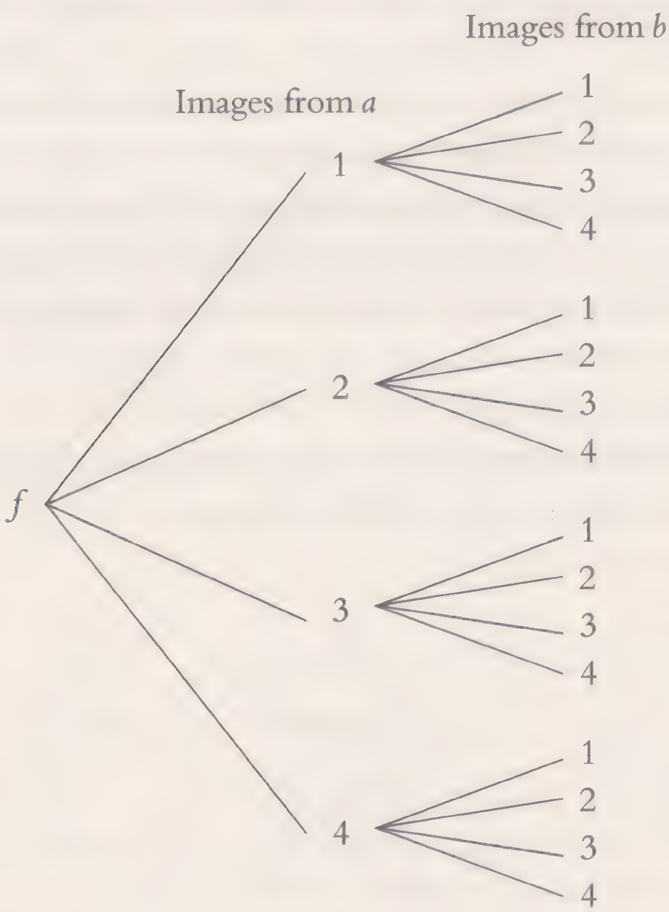


An exhaustive function.



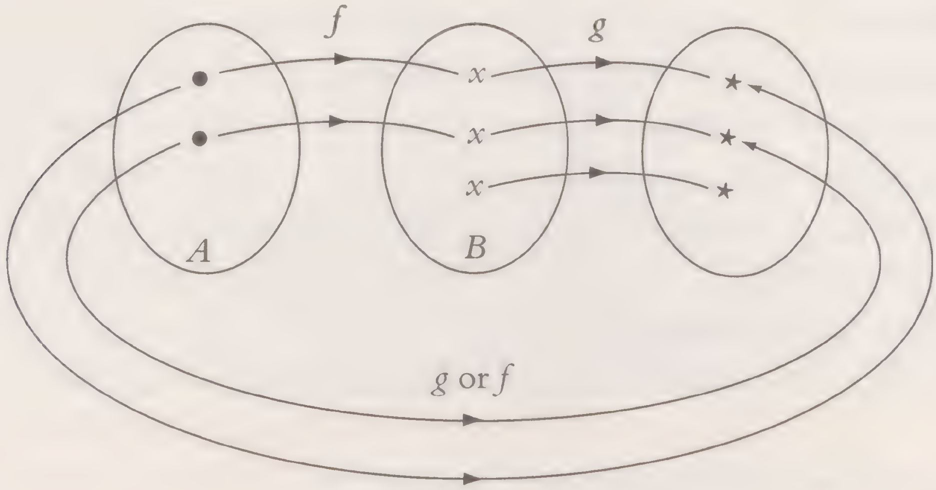
A bijective function.

In order to find all possible functions of a finite set A in another B it is useful to use graphs which are trees.



Tree of the possible functions of A = {a,b} in B = {1,2,3,4}.

In the case of having two functions f from A in B and g from B in C it makes sense to find the composition g or f of A in C in which $g(f(a))$ is assigned to all of a from A in C. So, in the finite case, there are composition graphs of the following style.

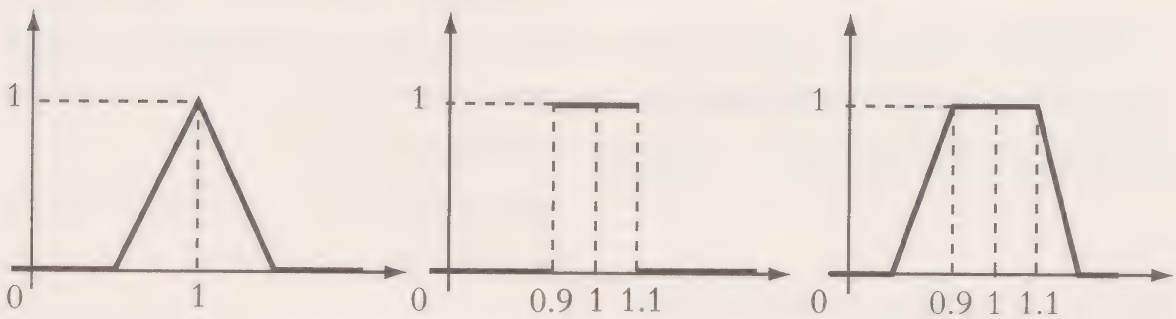


Composition graph of g with f.

Fuzzy sets and graphs

In recent decades and in order to model many complex real life situations *fuzzy set theory* has been developed with great success, founded by an engineer at the University of California (Berkeley), Lotfi Zadeh. In the classic approach, an element a belongs to or does not belong to a set A and therefore, that set can be identified by its characteristic function (with a value of 1 for elements of A and 0 for elements which do not belong to A).

Zadeh's idea was to expand the characteristic functions and construct *fuzzy sets*, in other words functions f associated with a set A in universe X which assign elements x from X with values $f(x)$ between 0 and 1 (real unit interval $[0,1]$) interpreting $f(x)$ as the *degree of membership* of x in A.



Fuzzy sets modelling 'the result is approximately 1'.

JOURNALS ON DISCRETE MATHEMATICS, COMBINATORICS AND GRAPHS

Currently the main magazines on these subjects are:

- *Ars Combinatorica.*
- *Combinatorica.*
- *Combinatorics, Probability and Computing.*
- *Designs, Codes and Cryptology.*
- *Discrete and Computational Geometry.*
- *Discrete Applied Mathematics.*
- *Discrete Mathematics.*
- *Electronic Journal of Combinatorics.*
- *European Journal of Combinatorics.*
- *Geombinatorics.*
- *Journal of Algebraic Combinatorics.*
- *Journal of Combinatorial Theory. Series A.*
- *Journal of Combinatorial Theory. Series B.*
- *Journal of Geometry.*
- *Journal of Graph Theory.*

There are many varied fuzzy models associated with one vague concept, and that is what makes the subject interesting, as it addresses different alternatives. Problems of artificial intelligence, machine control, digital photographs, image recognition, etc., even washing machines with fuzzy logic, are beautiful and useful examples of applications of this theory. Introducing *degrees* is a great idea. There is an infinite scale of greys between black and white.

The theory of fuzzy sets also includes the key subjects of making *fuzzy classifications and ordinations* and being able to introduce *degrees of relation*. This theory builds on theories of sets and could be rationalised with probability theory (where there are valuations between 0 and 1), but its interest lies in experimental models and providing solutions to problems which do not have a perfect and clear solution in the framework of mathematical models.

In particular, in the theory of fuzzy sets *graphs of relations* appear, but when values between 0 and 1 are assigned to pairs of related elements they are associated with the edges of the graph, in other words, they are *weighted graphs.*

With the comments in this section we hope to have demonstrated how graph theory also allows formulation in the theoretical framework of set theory and how, in mathematical visualisation itself, graphs can play an important role.

Glossary

Adjacent arcs Two arcs which have a vertex in common.

Adjacent edges Two edges which have a vertex in common.

Algorithm Step-by-step description of how to resolve a problem.

Arc Ordered pair of vertices, represented by an edge with an arrow.

Complete graph Graph in which all pairs of vertices are joined by a single edge of a graph.

Connected graph Graph in which a route can be found through the edges of the graph between any two vertices.

Circuit Path which starts and ends at the same vertex.

Critical path The longest path on a digraph with order requirements.

Degree of a vertex The number of edges of a graph which connect to the vertex.

Digraph Graph with edges directed by means of arrows, in other words, with arcs.

Directed graph Graph whose edges are all directed arcs.

Edge Link between two points (vertices) on a graph.

Eulerian circuit Circuit which passes through each edge of a graph exactly once.

Eulerian graph Graph with a Eulerian circuit.

Face Region limited by the edges of a graph.

Flow Amount of something associated with an edge, arc or graph.

Forest Set of tree graphs.

Graph Mathematical structure determined by *points* (or vertices) and *edges* (or lines) between some of these points.

Graph colouring Assigning colours to vertices, edges or faces of a graph complying with specific conditions.

Hamiltonian circuit Circuit which starts and ends at the same vertex passing through all other vertices just once via edges of the graph.

Hamiltonian graph Graph with a Hamiltonian circuit.

Homeomorphic graphs Two or more graphs such that one can be converted into the other by adding or removing vertices of degree 2 in its edges.

Isomorphic graphs Two or more graphs which have a two-way correspondence between the vertices and edges, in terms of both adjacency and degrees.

Label Information associated with the vertices and edges of a graph; the information can be numbers, words, measurements, names, etc.

Loop Arc or edge which starts and ends at the same vertex.

Matrix of a graph Ordered set of $n \times n$ numbers which are 1 or 0 and which correspond to the existing edges (1) or non-existing edges (0) between n vertices.

Node Vertex.

Organigram Graph which organises information, steps to be followed in a task or organisational aspects.

Network Graph used in transport and distribution.

Optimum solution. The best solution (in terms of any quantitative criterion) from a set of solutions.

Path Succession of adjacent arcs or edges.

Planar graph Graph in whose representation only the edges only intersect at the vertices.

Subgraph (of a graph). Graph determined by some of the vertices of the graph and some of the edges between them.

Tree Connected graph without circuits or cycles.

Route Path.

Vertex. Point of a graph which is isolated or where one or more edges end.

Weight Value assigned to the edge of a graph indicating cost, distance, time, etc.

Weighted graph Graph with numbers associated to vertices or edges.

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From Tube Maps to Neural Networks

The theory of graphs

A graph is an extraordinarily simple construct: a set of points joined by lines. Graphs include everything from underground maps to the delivery route taken by a postal worker. In fact, graphs can describe the multitude of networks that form the basis of our world. Carefully observing these simple structures opens our eyes to a universe of links and connections in which mathematics reigns supreme.